

INERTIAL GAME DYNAMICS AND APPLICATIONS TO CONSTRAINED OPTIMIZATION

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ABSTRACT. We derive a class of inertial dynamics for games and constrained optimization problems over simplices by building on the well-known “heavy ball with friction” method. In the single-agent case, the dynamics are generated by endowing the problem’s configuration space with a Hessian–Riemannian structure and then deriving the equations of motion for a particle moving under the influence of the problem’s objective (viewed as a potential field); for normal form games, the procedure is similar, but players are instead driven by the *unilateral* gradient of their payoff functions. By specifying an explicit Nash–Kuiper embedding of the simplex, we show that global solutions exist if and only if the interior of the simplex is mapped isometrically to a closed hypersurface of some ambient Euclidean space, and we characterize those Hessian–Riemannian structures which have this property. For such structures, low-energy solutions are attracted to isolated minimizers of the potential, showing in particular that pure Nash equilibria of potential games are attracting; more generally, when the game is not a potential one, we establish an inertial variant of the folk theorem of evolutionary game theory, showing that strict Nash equilibria attract all nearby strategy profiles.

1. INTRODUCTION

The vast majority of problems in mechanical control and classical mechanics are governed by the “heavy ball with friction” incarnation of Newton’s second law:

$$\ddot{x} = -\operatorname{grad} V - \eta \dot{x}, \quad (\text{HBF})$$

where the smooth function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ represents the system’s potential and $\eta \geq 0$ is a friction coefficient which dampens the system and controls the rate of energy dissipation. The physical principle of potential energy minimization then states that the stable – and, if $\eta > 0$, also attracting – states of (HBF) are the local minimizers of V , i.e. the states where the system cannot dissipate any more potential energy as heat.

Even though this principle is not a precise mathematical statement (and does not always hold either), treating (HBF) as a second order gradient-like system has given rise to an extensive literature in optimization – see e.g. [2, 3, 5, 7, 13, 25] and references therein. To wit, Attouch, Goudou & Redont showed in [7] that the

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inertial system (HBF) can be used for the global exploration of the local minima of V : by giving the system enough energy, one can escape the attraction of a local minimum and converge instead to the global minimum of V . What's more, if V is convex and bounded from below, Alvarez showed in [2] that the principle of energy minimization holds globally, i.e. the solutions of (HBF) converge to a minimizer of V from all initial conditions.

In this paper, our primary motivation is to extend these second order, inertial methods to mixed strategy learning in games, viewed here as concurrent, multi-agent, multi-objective optimization problems over products of simplices. The first difficulty in this endeavor is that (HBF) cannot be immediately constrained to a compact polytope X of \mathbb{R}^n , even if the driving force $F = -\text{grad } V$ vanishes on the boundary $\text{bd}(X)$ of X : if a solution trajectory of (HBF) starts with a sufficiently high velocity, then it will escape X in finite time. In view of this, Fl am and Morgan studied in [12] a projected variant of (HBF) where trajectories were constrained to X by projecting to zero any inadmissible velocity components at the boundary of X . However, as a result of this projection, the dynamics of [12] end up not being Lipschitz, so even existence and uniqueness results are hard to obtain.

To avoid well-posedness problems of this sort, we construct an interior-point learning method by endowing the problem's state space X with a Riemannian metric g which blows up at the boundary of X , and then deriving the associated covariant version of (HBF): the metric's blow-up will slow down trajectories near the boundary of X , so the resulting system is expected to be well-posed. In first order, this idea was exploited by Alvarez, Brahic & Bolte [4] who studied the gradient flow $\dot{x} = -\text{grad}_g V$ for a general class of Hessian–Riemannian metrics obtained by taking the Hessian $g = \text{Hess}(h)$ of a Legendre-type function h on X , i.e. a strictly convex function whose derivative blows up at the boundary of X : with $\text{grad}_g V$ vanishing on $\text{bd}(X)$, the corresponding gradient flow stays in X for all time and converges to a minimizer of V when V is convex [4].

To apply this Hessian–Riemannian approach to a second order optimization setting, the acceleration \ddot{x} in (HBF) must also be replaced by its covariant analogue, viz. the covariant derivative $D^2x/Dt^2 \equiv \nabla_{\dot{x}}\dot{x}$ of a curve's velocity along itself. To that end, we calculate the Levi-Civita connection of the Hessian–Riemannian structure on the simplex and derive an explicit expression for the covariant “heavy ball with friction” dynamics

$$\frac{D^2x}{Dt^2} = -\text{grad}_g V - \eta\dot{x}, \quad (\text{HBF}^*)$$

i.e. the equations of motion for a particle moving under the influence of the system's objective function (viewed as a potential field). These dynamics are then carried over to normal form games by replacing the gradient of the objective with the *unilateral* gradient of each player's individual payoff function, i.e. the gradient-like vector obtained by differentiating along the strategies of the focal player and keeping the strategies of his opponents fixed.

In the special case of the Shahshahani metric (corresponding to the Hessian of the Gibbs entropy) [28], this approach yields an inertial variant of the well-known replicator dynamics of evolutionary game theory [29].¹ The resulting dynamics

¹Similarly, in first order, the Hessian–Riemannian gradient flow corresponding to the Gibbs entropy function is just the ordinary replicator equation [4].

thus beg to be compared with the second order replicator equation that was recently derived in [17] as the offshoot of a higher order exponential learning scheme: therein, it was shown that even weakly dominated strategies become extinct and that strict Nash equilibria attract all nearby strategy profiles. On the other hand (and somewhat surprisingly at that), the inertial replicator dynamics exhibit no such properties: despite all intents to the contrary, their solution orbits collide with the boundary of X in finite time.

This phenomenon is due to the fact that the blow-up of the Shahshahani metric is not sufficient to contain trajectories within X . Indeed, as was shown by Akin [1], the n -dimensional simplex endowed with the Shahshahani metric is isometric to an (open) orthant of the n -dimensional sphere. This escape can thus be explained by noting that a particle which is constrained to move on the sphere under the influence of a finite force will escape any orthant of the sphere in finite time. Extending this reasoning, we derive an explicit Nash–Kuiper embedding of the simplex which allows us to characterize those Hessian–Riemannian structures for which (HBF) is defined for all time: if the interior of the game’s strategy space X can be mapped isometrically to a closed hypersurface of some ambient Euclidean space, then the trajectories of the covariant version of (HBF) will remain in X for all time.

When this is the case (the metric generated by the Hessian of the log-barrier function on the simplex being a prime example), we show that isolated minimizers of the potential attract all nearby solutions which start with low enough energy. On the other hand, as far as games are concerned, we establish in Section 5 the following inertial variant of the folk theorem of evolutionary game theory [15, 27] and we show that: *a*) Nash equilibria are stationary; *b*) if an interior orbit converges, its limit is a restricted equilibrium; *c*) if every neighborhood of a point admits an interior orbit which remains in the neighborhood in question for all time, this point is a restricted equilibrium; and, finally, *d*) strict equilibria attract all nearby strategy profiles.

Paper outline. In Section 2, we give a brief account of the geometric machinery that we will need along with some elementary definitions from game theory. Our analysis proper begins in Section 3 where we introduce a class of (decomposable) Hessian–Riemannian metrics with respect to which we derive the covariant analogue of (HBF) for the simplex; we also derive there an isometric embedding of the n -dimensional simplex in Euclidean space which allows us to view the derived dynamics as a classical mechanical system constrained to move on a hypersurface of \mathbb{R}^{n+1} . Section 4 is then devoted to single-agent, constrained optimization, while Section 5 contains our game-theoretic results; finally, in Section 6 we show that the inertial replicator dynamics do not admit global solutions (so the analysis of Sections 4 and 5 does not apply), and we characterize those Hessian–Riemannian structures that do.

2. NOTATION AND PRELIMINARIES

In this section our aim will be to fix our game-theoretic notation and to review some basic notions from Riemannian geometry, mostly following the excellent reference [18] for the latter.

2.1. Notational conventions. Let $\mathcal{S} = \{s_\alpha\}_{\alpha=0}^n$ be a finite set. The real vector space spanned by \mathcal{S} will be the set of formal linear combinations of elements of \mathcal{S} with real coefficients, i.e. the set $\mathbb{R}^{\mathcal{S}}$ of all maps $x: \mathcal{S} \rightarrow \mathbb{R}$, $s \in \mathcal{S} \mapsto x_s \in \mathbb{R}$. The canonical basis $\{e_s\}_{s \in \mathcal{S}}$ of this space consists of the indicator functions $e_s: \mathcal{S} \rightarrow \mathbb{R}$ which take the value $e_s(s) = 1$ on s and vanish otherwise, so thanks to the natural identification $s \mapsto e_s$, we will make no distinction between the elements s of \mathcal{S} and the corresponding basis vectors e_s of $\mathbb{R}^{\mathcal{S}}$. In fact, we will use α to refer interchangeably to either s_α or e_α , and the set $\Delta(\mathcal{S})$ of probability measures on \mathcal{S} will be identified with the n -dimensional simplex $\Delta(\mathcal{S}) \equiv \{x \in \mathbb{R}^{\mathcal{S}} : \sum_\alpha x_\alpha = 1 \text{ and } x_\alpha \geq 0\}$.

Regarding players and their actions, we will follow the original convention of Nash and employ Latin indices (k, ℓ, \dots) for players, while keeping Greek ones (α, β, \dots) for their actions (pure strategies); also, unless mentioned otherwise, we will use α, β, \dots for indices that start at 0, and κ, μ, ν, \dots (or Latin characters) for those which start at 1. Finally, if $\{\mathcal{S}_k\}_{k \in \mathcal{K}}$ is a family of finite sets, we will write $(\alpha; \alpha_{-k})$ for the tuple $(\alpha_1, \dots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \dots) \in \prod_k \mathcal{S}_k$, and we will use the shorthand $\sum_{\alpha \in \mathcal{S}_k}^k$ instead of $\sum_{\alpha \in \mathcal{S}_k}$.

2.2. Definitions from game theory. A *finite game* $\mathfrak{G} \equiv \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$ will be a tuple consisting of: a) a finite set of *players* $\mathcal{N} = \{1, \dots, N\}$; b) a finite set \mathcal{A}_k of *actions* (or *pure strategies*) for each player $k \in \mathcal{N}$; and c) the players' *payoff functions* $u_k: \mathcal{A} \rightarrow \mathbb{R}$, where $\mathcal{A} \equiv \prod_k \mathcal{A}_k$ denotes the game's *action space*, i.e. the set of all *action profiles* $(\alpha_1, \dots, \alpha_N)$, $\alpha_k \in \mathcal{A}_k$.

If players mix their actions by taking probability distributions $x_k = (x_{k\alpha})_{\alpha \in \mathcal{A}_k} \in \Delta(\mathcal{A}_k)$ over their action sets \mathcal{A}_k , their expected payoffs will be

$$u_k(x) = \sum_{\alpha_1}^1 \cdots \sum_{\alpha_N}^N u_k(\alpha_1, \dots, \alpha_N) x_{1,\alpha_1} \cdots x_{N,\alpha_N}, \quad (2.1)$$

where $x = (x_1, \dots, x_N)$ is the players' (mixed) *strategy profile* and $u_k(\alpha_1, \dots, \alpha_N)$ denotes the payoff to player k in the (pure) action profile $(\alpha_1, \dots, \alpha_N) \in \mathcal{A}$; moreover, if player k plays the pure strategy $\alpha \in \mathcal{A}_k$, we will use the notation $u_{k\alpha}(x) \equiv u_k(\alpha; x_{-k}) = u_k(x_1, \dots, \alpha, \dots, x_N)$. In this mixed context, the *strategy space* of player k will be the simplex $X_k \equiv \Delta(\mathcal{A}_k)$ while the strategy space of the game will be the convex polytope $X \equiv \prod_k X_k$. Together with the players' (expected) payoff functions $u_k: X \rightarrow \mathbb{R}$, the tuple (\mathcal{N}, X, u) will be called the *mixed extension* of \mathfrak{G} and it will also be denoted by \mathfrak{G} (relying on context to resolve any ambiguities).

The most prominent solution concept in game theory is the concept of Nash equilibrium which characterizes strategy profiles that are resilient against unilateral deviations. More formally, we will say that $q \in X$ is a Nash equilibrium of \mathfrak{G} when

$$u_k(x_k; q_{-k}) \leq u_k(q) \quad \text{for all } x_k \in X_k \text{ and for all } k \in \mathcal{N}, \quad (\text{NE})$$

and if the above inequality is strict for all $x_k \in X_k \setminus \{q_k\}$, $k \in \mathcal{N}$, then q will be called a *strict* Nash equilibrium – obviously, by multilinearity, only “pure” strategy profiles (viz. the vertices of X) can be strict equilibria. On the other hand, if (NE) only holds for x_k in the subface X'_k spanned by the support $\text{supp}(q_k) = \{\alpha \in \mathcal{A}_k : q_{k\alpha} > 0\}$ of q_k , then q will be called a *restricted Nash equilibrium* of \mathfrak{G} , reflecting the fact that it is a Nash equilibrium of the restriction $\mathfrak{G}' \equiv \mathfrak{G}|_{X'} = \mathfrak{G}'(\mathcal{N}, X', u|_{X'})$ of \mathfrak{G} to X' .

An especially relevant class of finite games is obtained when the players' payoff functions satisfy the *potential property*:

$$u_{k\alpha}(x) - u_{k\beta}(x) = -(V(\alpha; x_{-k}) - V(\beta; x_{-k})) \quad (2.2)$$

for some (necessarily) multilinear function $V: X \rightarrow \mathbb{R}$. When this is the case, the game will be called a *potential game with potential function V* , and as is well known, the pure Nash equilibria of \mathfrak{G} will be the vertices of X that are local minimizers of V [23].

2.3. Elements of Riemannian geometry. For notation and as a general reference on Riemannian geometry, we refer the reader to the masterful account of [18]; in what follows, we simply give a brief overview of the notions that we will use for clarity and completeness.

Let U be an open set of \mathbb{R}^m and let $p \in U$. The *tangent space* to U at p is traditionally defined as the space $\mathbb{R}_p^m = \{p\} \times \mathbb{R}^m = \{(p, z) : z \in \mathbb{R}^m\}$; equivalently, a tangent vector $X \in T_p U$ may also be seen as a linear operator acting on smooth functions $f \in C^\infty(U)$ by means of directional differentiation, i.e. as a map $f \mapsto X(f) \equiv \left. \frac{d}{dt} \right|_{t=0} f(p + tX)$. Dropping the basepoint p for notational convenience, the standard basis of $T_p U$ will be denoted as $\{E_j\}_{j=1}^m$, and, depending on the context, E_j will be viewed interchangeably as a column vector in \mathbb{R}^m or as an operator sending $f \mapsto E_j(f) \equiv \partial_j f|_p$.

The *cotangent space* to U at p will be the dual space $T_p^* U \equiv (T_p U)^*$ of $T_p U$, and the action of a *covector* (or *1-form*) $\omega \in T_p^* U$ on $X \in T_p U$ will be written in Dirac's bra-ket notation as $\omega(X) \equiv \langle \omega | X \rangle$. Fiberizing the above constructions over U , we then obtain the *tangent* (resp. *cotangent*) *bundle* $TU \equiv \coprod_{p \in U} T_p U$ (resp. $T^*U \equiv \coprod_{p \in U} T_p^* U$); also, the space of vector (resp. covector) fields on U will be denoted by $\mathcal{T}(U)$ (resp. $\mathcal{T}^*(U)$).

An *inner product* on $T_p U$ will be a positive-definite pairing on $T_p U$, i.e. a bilinear form $g_p: T_p U \times T_p U \rightarrow \mathbb{R}$ such that:

- (1) $g_p(X, Y) = g_p(Y, X)$ for all $X, Y \in T_p U$.
- (2) $g_p(X, X) \geq 0$ with equality if and only if $X = 0$.

If there is no doubt as to the metric tensor g in question, we will occasionally use the more suggestive notation $\langle X, Y \rangle_p \equiv g_p(X, Y)$; in the same spirit, when the basepoint $p \in U$ may be inferred from the context, we will drop it altogether and simply write $\langle X, Y \rangle$ instead. The norm of X is then defined as

$$\|X\|^2 = \langle X, X \rangle \quad (2.3)$$

and the orthogonal projection of $X \in T_p U$ on $Y \in T_p U$ will be

$$\text{pr}_Y X = \frac{\langle X, Y \rangle}{\|Y\|^2} Y. \quad (2.4)$$

Extending these constructions to U , a *Riemannian metric* will be a smooth assignment of an inner product g_p to every $p \in U$. The components of g in the frame $\{E_j\}$ will then be

$$g_{ij}(p) = g_p(E_i, E_j), \quad (2.5)$$

so a Riemannian metric may be seen as a smooth field of positive-definite matrices on U . From a dual point of view, given that the pairing $g_p: T_p U \times T_p U \rightarrow \mathbb{R}$

is positive-definite (and, hence, non-singular), we may assign to each covector $\omega \in T_p^*U$ a unique vector $\omega^\sharp \in T_pU$ such that

$$\langle \omega^\sharp, Y \rangle = \langle \omega, Y \rangle \quad \text{for all } Y \in T_pU. \quad (2.6)$$

The resulting tangent-cotangent isomorphism $\sharp: T_p^*U \rightarrow T_pU$ is known colloquially as the “musical isomorphism”, and it may be written in components as

$$\omega_j^\sharp = \sum_{k=1}^m g^{jk} \omega_k, \quad (2.7)$$

where g^{jk} denotes the inverse matrix of g , i.e.

$$\sum_{j=1}^n g_{ij} g^{jk} = \delta_{ik}. \quad (2.8)$$

Accordingly, we define the gradient of a smooth function $f \in C^\infty(U)$ with respect to g as

$$\text{grad}_g f = (df)^\sharp, \quad (2.9)$$

or, in components:

$$(\text{grad}_g f)_j = \sum_{k=1}^m g^{jk} \frac{\partial f}{\partial x_k}. \quad (2.10)$$

As usual, when there is no danger of confusion, $\text{grad}_g f$ will be denoted more simply by $\text{grad } f$.

In view of the above, differentiating a function $f \in C^\infty(U)$ along a vector field $X \in \mathcal{T}(U)$ simply amounts to taking the pairing $\langle \text{grad } f, X \rangle \equiv \langle df|X \rangle \equiv X(f)$. On the other hand, to differentiate a vector field along another, we will need the notion of a (linear) *connection* on U , viz. a map

$$\nabla: \mathcal{T}(U) \times \mathcal{T}(U) \rightarrow \mathcal{T}(U) \quad (2.11)$$

written $(X, Y) \mapsto \nabla_X Y$, and such that:

- (1) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$ for all $f_1, f_2 \in C^\infty(U)$.
- (2) $\nabla_X (aY_1 + bY_2) = a \nabla_X Y_1 + b \nabla_X Y_2$ for all $a, b \in \mathbb{R}$.
- (3) $\nabla_X (fY) = f \cdot \nabla_X Y + \nabla_X f \cdot Y$ for all $f \in C^\infty(U)$, where $\nabla_X f \equiv X(f) = \langle df|X \rangle$.

In this way, $\nabla_X Y$ generalizes the idea of differentiating Y along X and it will be called the *covariant derivative of Y in the direction of X* .

In the standard frame $\{E_j\}$ of TU , the defining properties of ∇ give

$$\nabla_X Y = \sum_{j,k=1}^n X_j \frac{\partial Y_k}{\partial x_j} E_k + \sum_{i,j,k=1}^n \Gamma_{ij}^k X_i Y_j E_k, \quad (2.13)$$

where the *Christoffel symbols* $\Gamma_{ij}^k \in C^\infty(U)$ of ∇ in the frame $\{E_j\}$ are defined via the equation

$$\nabla_{E_i} E_j = \sum_{k=1}^n \Gamma_{ij}^k E_k. \quad (2.14)$$

Clearly, ∇ is completely specified by its Christoffel symbols, so there is no canonical connection on U ; however, if U is also endowed with a Riemannian metric g , then there exists a *unique* connection which is *symmetric* (i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$) and *compatible* with g in the sense that:

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \quad \text{for all } X, Y, Z \in \mathcal{T}(U). \quad (2.15)$$

This connection is known as the *Levi-Civita connection* on U , and its Christoffel symbols are given in coordinates by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^n g^{k\ell} \left(\frac{\partial g_{\ell j}}{\partial x_i} + \frac{\partial g_{\ell i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell} \right). \quad (2.16)$$

In view of the above, the *covariant derivative of V along a curve γ on U* will be

$$\frac{DV}{Dt} \equiv \nabla_{\dot{\gamma}} V \equiv \sum_{i,j,k=1}^n \left(\dot{V}_k + \Gamma_{ij}^k V_i \dot{\gamma}_j \right) E_k \quad (2.17)$$

so, specializing to the case where $V(t)$ is simply the *velocity* $v(t) = \dot{\gamma}(t)$ of γ , the *acceleration* of γ will be defined as $\frac{D^2\gamma}{Dt^2} = \frac{Dv}{Dt} = \nabla_{\dot{\gamma}} \dot{\gamma}$ or, in components:

$$\frac{D^2\gamma_k}{Dt^2} \equiv \ddot{\gamma}_k + \sum_{i,j} \Gamma_{ij}^k \dot{\gamma}_i \dot{\gamma}_j. \quad (2.18)$$

3. DERIVATION OF THE DYNAMICS

Motivated by the optimization properties of gradient-like dynamics that are second order in time [2, 3], our goal in this section will be to derive a covariant version of (HBF) for the n -dimensional simplex $X \equiv \Delta(n+1)$ of \mathbb{R}^{n+1} . Of course, if X is endowed with the standard Euclidean metric (inherited from the ambient space \mathbb{R}^{n+1} in which it resides), then there will be no constraint force restricting the dynamics to remain in the simplex and solution orbits will escape in finite time. As such, we will instead follow the Hessian–Riemannian approach of [4] and try to raise an inherent geometric barrier at the boundary of X by endowing the positive orthant $\mathbb{R}_{++}^{n+1} \equiv \{x \in \mathbb{R}^{n+1}, x_j > 0\}$ of \mathbb{R}^{n+1} with a Riemannian metric which blows up at the boundary hyperplanes $x_\alpha = 0$.

3.1. Hessian–Riemannian metrics. In the context of [4], this blow-up property is obtained by letting the metric tensor be the Hessian of an entropy-like functional that becomes infinitely steep near the boundary of \mathbb{R}_{++}^{n+1} . Specifically, let $\theta: [0, +\infty) \rightarrow \mathbb{R} \cup \{+\infty\}$ be a C^∞ function satisfying the Legendre-type properties [4, 26]:²

1. $\theta(x) < \infty$ for all $x > 0$.
 2. $\lim_{x \rightarrow 0^+} \theta'(x) = -\infty$.
 3. $\theta''(x) > 0$ and $\theta'''(x) < 0$ for all $x > 0$.
- (L)

Moreover, let

$$h(x) = \sum_{\alpha=0}^n \theta(x_\alpha) \quad (3.1)$$

and define a metric g on \mathbb{R}_{++}^{n+1} by taking the Hessian of h in standard coordinates, i.e.:

$$g_{\alpha\beta} = \frac{\partial^2 h}{\partial x_\alpha \partial x_\beta} = \theta''_\alpha \delta_{\alpha\beta}, \quad (3.2)$$

²Legendre-type functions are usually defined without the regularity requirement $\theta''' < 0$. This assumption can be relaxed without significantly affecting our results but we will keep it as is to simplify our presentation.

where the shorthand θ''_α , $\alpha = 0, \dots, n$, stands for $\theta''(x_\alpha)$. In other words, the *Hessian–Riemannian metric associated to θ* will be the field of positive-definite matrices

$$g(x) = \text{diag}(\theta''(x_0), \dots, \theta''(x_n)), \quad x \in \mathbb{R}_{++}^{n+1}. \quad (3.3)$$

With θ strictly convex ($\theta''(x) > 0$), it follows that g is indeed a Riemannian metric on \mathbb{R}_{++}^{n+1} ; accordingly, when g is derived in this way, θ will be called the *kernel* of g and h will be its associated *entropy function*.

Remark 3.1. The Legendre-type function h of (3.1) is closely related to the control cost functions used to define quantal responses in the theory of discrete choice [21, 30], so there is a deep connection between the geometry of the simplex and the theory of quantal/smoothed best responses; for an extended discussion of this issue, see [10, 22] and references therein.

Remark 3.2. More general Hessian–Riemannian structures can be obtained by considering Legendre-type functions $h: \mathbb{R}_{++}^{n+1} \rightarrow \mathbb{R}$ that do not necessarily admit a decomposition of the form $h(x) = \sum_{\alpha=0}^n \theta(x_\alpha)$. Most of our analysis can be extended to this more general setting, but since the calculations involved are significantly more tedious, we have opted to present our analysis in this simpler, decomposable setting.³

Example 1 (The Shahshahani metric). Perhaps the most well-known example of a Hessian–Riemannian structure on the simplex corresponds to the kernel $\theta_S(x) = x \log x$ which gives the *Shahshahani metric* [1, 4, 28]

$$g^S(x) = \text{diag}(1/x_0, \dots, 1/x_n), \quad x \in \mathbb{R}_{++}^{n+1}, \quad (3.4)$$

or, in coordinates:

$$g_{\alpha\beta}^S = \delta_{\alpha\beta} / x_\beta, \quad (3.5)$$

As the Hessian of the (negative of the) Gibbs entropy $h(x) = \sum_\alpha x_\alpha \log x_\alpha$,⁴ the Shahshahani metric is intimately tied to the well-known replicator dynamics of evolutionary game theory [4, 14, 15], so a large part of our analysis will be focused on g^S and its properties – see [14, 15] and [27] for an excellent discussion on the topic.

Example 2 (The log-barrier). Another noteworthy example with important ties to proximal methods in optimization (see [4] and references therein) is given by the logarithmic barrier kernel $\theta_L(x) = -\log x$ which generates the well-known Itakura-Saito divergence [4, 8, 11, 20]. This choice of θ corresponds to the log-barrier function $h(x) = -\sum_\alpha \log x_\alpha$ whose Hessian generates the metric

$$g_{\alpha\beta}^L = \delta_{\alpha\beta} / x_\beta^2, \quad (3.6)$$

or, in matrix form:

$$g^L(x) = \text{diag}(1/x_0^2, \dots, 1/x_n^2), \quad x \in \mathbb{R}_{++}^{n+1}. \quad (3.7)$$

An important qualitative difference between g^S and g^L is that the kernel of the former is finite everywhere, whereas the kernel of the latter blows up at 0; as we

³In particular, the results that do not hold verbatim are those that call for an explicit expression on θ – most notably, Corollary 6.2. The analysis of Section 3.4 will also carry through by invoking the Nash–Kuiper embedding theorem – all that is lost is the explicit expression (EC).

⁴This is also the reason that we are calling h the entropy function of g .

shall see later, this point will have important consequences with regards to the existence of global solutions.

Now, having endowed \mathbb{R}_{++}^{n+1} with a Riemannian metric g , the interior X° of the unit n -dimensional simplex Δ inherits itself a Riemannian structure by restricting g to TX° . For calculations (and to make this idea more explicit), it will be convenient to introduce (global) coordinates on X° via the surjection $\pi_0: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ which forgets the 0-th component of (x_0, x_1, \dots, x_n) , i.e.:

$$\pi_0(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n). \quad (3.8)$$

In this way, if we denote the image of X° under π_0 by $U = \{w \in \mathbb{R}^n : w_\mu > 0 \text{ and } \sum_{\mu=1}^n w_\mu < 1\}$, the injective immersion $\iota_0: U \rightarrow X^\circ$ defined as

$$\iota_0(w_1, \dots, w_n) = (1 - \sum_{\mu=1}^n w_\mu, w_1, \dots, w_n) \quad (3.9)$$

will be an inverse to $\pi_0|_{X^\circ}$, and hence a smooth embedding. The standard coordinate frame $\{\tilde{E}_\mu\}_{\mu=1}^n$ of TU corresponding to the coordinates (w_1, \dots, w_n) of \mathbb{R}^n will thus push forward via ι_0 to the frame $E_\mu - E_0$ of TX° in the sense that

$$\frac{\partial}{\partial w_\mu}(f \circ \iota_0) = \frac{\partial f}{\partial x_\mu} - \frac{\partial f}{\partial x_0} \quad \text{for all } f \in C^\infty(\mathbb{R}^{n+1}) \quad (3.10)$$

As such, the induced Riemannian metric \tilde{g} on U may be expressed in components as:

$$\tilde{g}_{\mu\nu} \equiv \tilde{g}(\tilde{E}_\mu, \tilde{E}_\nu) = g(E_\mu - E_0, E_\nu - E_0) = \theta''_\mu \delta_{\mu\nu} + \theta''_0, \quad (3.11)$$

where $\theta''_\beta(x) = \theta''(x_\beta)$ and θ''_0 is to be understood as $\theta''_0(w) = \theta''(1 - \sum_\mu w_\mu)$.

3.2. Covariant differentiation and Newton's law. With this coordinate representation at hand, the Christoffel symbols $\tilde{\Gamma}_{\mu\nu}^\kappa$ of \tilde{g} will be given by the expression $\tilde{\Gamma}_{\mu\nu}^\kappa = \sum_\rho \tilde{g}^{\kappa\rho} \tilde{\Gamma}_{\rho\mu\nu}$ where, following (2.16), the Christoffel symbols of the first kind $\tilde{\Gamma}_{\rho\mu\nu}$ are defined as:

$$\tilde{\Gamma}_{\rho\mu\nu} = \frac{1}{2} \left(\frac{\partial \tilde{g}_{\rho\mu}}{\partial w_\nu} + \frac{\partial \tilde{g}_{\rho\nu}}{\partial w_\mu} - \frac{\partial \tilde{g}_{\mu\nu}}{\partial w_\rho} \right). \quad (3.12)$$

Even though a direct calculation of $\tilde{\Gamma}_{\rho\mu\nu}$ from (3.11) is possible if one is armed with enough patience, it is simpler to note that the push-forward expression (3.10) implies that $\tilde{g}_{\mu\nu} = \frac{\partial^2 \tilde{h}}{\partial w_\mu \partial w_\nu}$ where $\tilde{h} = h \circ \iota_0: U \rightarrow \mathbb{R}$ is the pull-back of h to U via ι_0 . By the equality of mixed partials, we thus obtain:

$$\tilde{\Gamma}_{\rho\mu\nu} = \frac{1}{2} \frac{\partial^3 \tilde{h}}{\partial w_\rho \partial w_\mu \partial w_\nu} = \frac{1}{2} \left(\theta'''_\rho \delta_{\rho\mu\nu} - \theta'''_0 \right), \quad (3.13)$$

where now $\delta_{\rho\mu\nu} = \delta_{\rho\mu} \delta_{\mu\nu}$ denotes the triangular Kronecker symbol ($\delta_{\rho\mu\nu} = 1$ if $\rho = \mu = \nu$ and 0 otherwise) and $\theta'''_\beta, \beta = 0, 1, \dots, n$, is shorthand for $\theta'''_\beta(x) = \theta'''(x_\beta)$.

To continue, we will need to calculate the inverse of the metric tensor $\tilde{g}_{\mu\nu} = \theta''_\mu \delta_{\mu\nu} + \theta''_0$ on U ; to that end, we have the following recipe:

Lemma 3.1. *Let $A_{\mu\nu} = q_\mu \delta_{\mu\nu} + q_0$ with $q_0, q_1, \dots, q_n > 0$. Then, the inverse $A^{\mu\nu}$ of $A_{\mu\nu}$ is*

$$A^{\mu\nu} = \frac{\delta_{\mu\nu}}{q_\mu} - \frac{Q_h}{q_\mu q_\nu}, \quad (3.14)$$

where Q_h denotes the harmonic aggregate $Q_h^{-1} \equiv \sum_{\alpha=0}^n q_\alpha^{-1}$.

Proof. By a straightforward verification, we have:

$$\begin{aligned} \sum_{\nu=1}^n A_{\mu\nu} A^{\nu\rho} &= \sum_{\nu} (q_\mu \delta_{\mu\nu} + q_0) (\delta_{\nu\rho} / q_\nu - Q_h / (q_\nu q_\rho)) \\ &= \sum_{\nu} (q_\mu \delta_{\mu\nu} \delta_{\nu\rho} / q_\nu + q_0 \delta_{\nu\rho} / q_\nu - q_\mu Q_h \delta_{\mu\nu} / (q_\nu q_\rho) - q_0 Q_h / (q_\nu q_\rho)) \\ &= \delta_{\mu\rho} + q_0 q_\rho^{-1} - Q_h q_\rho^{-1} - q_0 Q_h q_\rho^{-1} \sum_{\nu} q_\nu^{-1} = \delta_{\mu\rho}. \quad \square \end{aligned}$$

As a result, by setting $\Theta_h'' = \left(\sum_{\beta} 1 / \theta_\beta'' \right)^{-1}$, the inversion formula (3.14) above yields

$$\tilde{g}^{\mu\nu} = \frac{\delta_{\mu\nu}}{\theta_\mu''} - \frac{\Theta_h''}{\theta_\mu'' \theta_\nu''}, \quad (3.15)$$

and, in combination with (3.13) and (2.16), we finally obtain:

$$\begin{aligned} \tilde{\Gamma}_{\mu\nu}^\kappa &= \sum_{\rho=1}^n \tilde{g}^{\kappa\rho} \tilde{\Gamma}_{\rho\mu\nu} = \frac{1}{2} \sum_{\rho} \left(\frac{\delta_{\kappa\rho}}{\theta_\rho''} - \frac{\Theta_h''}{\theta_\rho'' \theta_k''} \right) (\theta_\rho''' \delta_{\rho\mu\nu} - \theta_0''') \\ &= \frac{1}{2} \left(\delta_{\kappa\mu\nu} \frac{\theta_\kappa'''}{\theta_\kappa''} - \frac{\Theta_h'' \theta_\mu'''}{\theta_\kappa'' \theta_\mu''} \delta_{\mu\nu} - \frac{\theta_0'''}{\theta_\kappa''} + \frac{\Theta_h'' \theta_0'''}{\theta_\kappa''} \left(\frac{1}{\Theta_h''} - \frac{1}{\theta_0''} \right) \right) \\ &= \frac{1}{2} \left(\delta_{\kappa\mu\nu} \frac{\theta_\kappa'''}{\theta_\kappa''} - \frac{\Theta_h'' \theta_\mu'''}{\theta_\kappa'' \theta_\mu''} \delta_{\mu\nu} - \frac{\theta_0'''}{\theta_0'' \theta_\kappa''} \right), \quad (3.16) \end{aligned}$$

where we used the fact that $\sum_{\rho=1}^n 1 / \theta_\rho'' = 1 / \Theta_h'' - 1 / \theta_0''$ in the second line. Consequently, the covariant acceleration (2.18) of a curve $w(t)$ on U will be:

$$\begin{aligned} \frac{D^2 w_\kappa}{Dt^2} &= \ddot{w}_\kappa + \frac{1}{2} \sum_{\mu,\nu} \left(\delta_{\kappa\mu\nu} \frac{\theta_\kappa'''}{\theta_\kappa''} - \frac{\Theta_h'' \theta_\mu'''}{\theta_\kappa'' \theta_\mu''} \delta_{\mu\nu} - \frac{\theta_0'''}{\theta_0'' \theta_\kappa''} \right) \dot{w}_\mu \dot{w}_\nu \\ &= \ddot{w}_\kappa + \frac{1}{2} \frac{\theta_\kappa'''}{\theta_\kappa''} \dot{w}_\kappa^2 - \frac{1}{2} \frac{\Theta_h''}{\theta_\kappa''} \sum_{\mu} \frac{\theta_\mu'''}{\theta_\mu''} \dot{w}_\mu^2 - \frac{1}{2} \frac{\theta_0'''}{\theta_0'' \theta_\kappa''} \left(\sum_{\mu} \dot{w}_\mu \right)^2. \quad (3.17) \end{aligned}$$

Despite the apparent ugliness of this last expression, things become considerably simpler if we write (3.17) in the original coordinates (x_0, x_1, \dots, x_n) of \mathbb{R}_{++}^{n+1} – i.e. by considering the curve $x(t) = \iota_0(w(t))$. In that case, with $x_0 = 1 - \sum_{\mu} w_\mu$, we will have $\sum_{\mu} \dot{w}_\mu = -\dot{x}_0$, and after some easy algebra, (3.17) gives the following expression for the covariant acceleration:

$$\frac{D^2 x_\alpha}{Dt^2} = \ddot{x}_\alpha + \frac{1}{2} \frac{1}{\theta_\alpha''} \left(\theta_\alpha''' \dot{x}_\alpha^2 - \sum_{\beta=0}^n \left(\Theta_h'' / \theta_\beta'' \right) \theta_\beta''' \dot{x}_\beta^2 \right). \quad (3.18)$$

Remark 3.3. The cumbersome calculations leading to (3.18) could have been circumvented by working directly on \mathbb{R}_{++}^{n+1} (where the Christoffel symbols of g are trivial to calculate), and then “projecting” the resulting second order expressions to X by employing the theory of holonomically constrained mechanical systems [6, 19] to the equality constraint $\sum_{\beta} x_\beta = 1$. However, the calculation of the resulting constraint force (the summation term in (3.18)) turns out to require a comparable amount of work by itself, and since this approach would necessitate the introduction of further machinery anyway, we chose to follow the more straightforward (albeit perhaps less elegant) approach above.

Now, to write down Newton's second law of motion for a frictionless heavy ball moving in X , it will again be more convenient to work with the variables w and then pull back the results to X° via ι_0 . Accordingly, let $\tilde{F} \in \mathcal{T}(U)$ be a force field on U written in components as $\tilde{F} = \sum_{\mu=1}^n \tilde{F}_\mu \tilde{E}_\mu$ where \tilde{E}_μ denotes the standard frame of U . Then, by virtue of the push-forward relation (3.10), the equations of motion in the standard coordinates of X° will take the form:

$$\frac{D^2 x_\alpha}{Dt^2} = \begin{cases} \tilde{F}_\alpha & \text{if } \alpha \neq 0, \\ -\sum_\mu \tilde{F}_\mu & \text{if } \alpha = 0. \end{cases} \quad (3.19)$$

Even though the direction $\alpha = 0$ is not really being discriminated against in the above equation, the resulting expression for Newton's law is not particularly handy to work with; on the other hand, things become considerably simpler and more intuitive in the conservative case where $F = -\text{grad } \tilde{V}$ for some smooth function $\tilde{V}: U \rightarrow \mathbb{R}$ corresponding to a potential $V = \tilde{V} \circ \pi_0: X^\circ \rightarrow \mathbb{R}$ on the simplex. In that case, the expression (3.15) for $\tilde{g}^{\mu\nu}$ readily yields:

$$\begin{aligned} \tilde{F}_\mu &= -\frac{1}{\theta''_\mu} \left(\frac{\partial \tilde{V}}{\partial w_\mu} - \sum_{\nu=1}^n \frac{\Theta''_h}{\theta''_\nu} \frac{\partial \tilde{V}}{\partial w_\nu} \right) \\ &= -\frac{1}{\theta''_\mu} \frac{\partial V}{\partial x_\mu} + \frac{1}{\theta''_\mu} \sum_{\nu=1}^n \frac{\Theta''_h}{\theta''_\nu} \frac{\partial V}{\partial x_\nu} + \frac{1}{\theta''_\mu} \frac{\Theta''_h}{\theta''_0} \frac{\partial V}{\partial x_0} = -\frac{1}{\theta''_\mu} \left(\frac{\partial V}{\partial x_\mu} - \sum_{\beta=0}^n \frac{\Theta''_h}{\theta''_\beta} \frac{\partial V}{\partial x_\beta} \right), \end{aligned} \quad (3.20a)$$

where in the second equality we used the fact that $\sum_\nu 1/\theta''_\nu = 1/\Theta''_h - 1/\theta''_0$; with some more algebra, we then get the following expression for the last term of (3.19):

$$\begin{aligned} -\sum_{\mu=1}^n \tilde{F}_\mu &= \sum_{\mu=1}^n \frac{1}{\theta''_\mu} \frac{\partial V}{\partial x_\mu} - \left(\frac{1}{\Theta''_h} - \frac{1}{\theta''_0} \right) \sum_{\beta=0}^n \frac{\Theta''_h}{\theta''_\beta} \frac{\partial V}{\partial x_\beta} \\ &= -\frac{1}{\theta''_0} \left(\frac{\partial V}{\partial x_0} - \sum_{\beta=0}^n \frac{\Theta''_h}{\theta''_\beta} \frac{\partial V}{\partial x_\beta} \right). \end{aligned} \quad (3.20b)$$

In view of the above, we may now write down an explicit expression for the Hessian–Riemannian analogue of (HBF) for the simplex. Indeed, starting with the equation of motion

$$\frac{D^2 x}{Dt^2} = -\text{grad } V - \eta \dot{x}, \quad (\text{HBF}^*)$$

and plugging in the covariant acceleration $D^2 x / Dt^2$ from (3.18) and the gradient of V from (3.20), we readily obtain the component-wise *inertial dynamics*:

$$\ddot{x}_\alpha = \frac{1}{\theta''_\alpha} \left(u_\alpha - \sum_\beta \left(\Theta''_h / \theta''_\beta \right) u_\beta \right) - \frac{1}{2} \frac{1}{\theta''_\alpha} \left(\theta'''_\alpha \dot{x}_\alpha^2 - \sum_\beta \left(\Theta''_h / \theta''_\beta \right) \theta'''_\beta \dot{x}_\beta^2 \right) - \eta \dot{x}_\alpha, \quad (\text{ID})$$

where we have set $u_\alpha = -\partial_\alpha V$ and the term $-\eta \dot{x}$ describes a friction force proportional to the particle's velocity.

The dynamics (ID) will constitute our main focus, so a few examples are in order:

Example 3 (The inertial replicator dynamics). In the case of the Shahshahani kernel $\theta(x) = x \log x$, we will have $\theta''(x) = 1/x$ and $\theta'''(x) = -1/x^2$, so (ID) yields the *inertial replicator dynamics*:

$$\dot{x}_\alpha = x_\alpha \left(u_\alpha - \sum_\beta x_\beta u_\beta \right) + \frac{1}{2} x_\alpha \left(\dot{x}_\alpha^2 / x_\alpha^2 - \sum_\beta \dot{x}_\beta^2 / x_\beta \right) - \eta \dot{x}_\alpha. \quad (\text{I-RD})$$

It is thus interesting to compare this inertial variant of the replicator dynamics in the frictionless case ($\eta = 0$) with the second order system

$$\ddot{x}_\alpha = x_\alpha \left(u_\alpha - \sum_\beta x_\beta u_\beta \right) + x_\alpha \left(\dot{x}_\alpha^2 / x_\alpha^2 - \sum_\beta \dot{x}_\beta^2 / x_\beta \right), \quad (\text{2-RD})$$

which appeared in [17] as an offshoot of the second order exponential learning scheme:

$$\begin{aligned} \ddot{y}_\alpha &= u_\alpha \\ x_\alpha &= \frac{\exp(y_\alpha)}{\sum_\beta \exp(y_\beta)}. \end{aligned} \quad (3.21)$$

Remarkably, despite their very different origins (from geometrical and learning considerations respectively), the only difference between these two second order extensions of the replicator dynamics is the factor $1/2$ in the “constraint force” term. Even more surprisingly, and in spite of the innocuous appearance of this scaling-like difference,⁵ we shall see in the following sections that (I-RD) and (2-RD) behave in drastically different ways.

Example 4. In the case of the log-barrier kernel $\theta(x) = -\log x$, we will have $\theta''(x) = 1/x^2$ and $\theta'''(x) = -2/x^3$, thus obtaining the *inertial log-barrier dynamics*:

$$\ddot{x}_\alpha = x_\alpha^2 \left(u_\alpha - r^{-2} \sum_\beta x_\beta^2 u_\beta \right) + x_\alpha^2 \left(\dot{x}_\alpha^2 / x_\alpha^3 - r^{-2} \sum_\beta \dot{x}_\beta^2 / x_\beta \right) - \eta \dot{x}_\alpha, \quad (\text{I-LD})$$

where $r^2 = \sum_{\beta=0}^n x_\beta^2$. The first order analogue of these dynamics, namely the system $\dot{x}_\alpha = x_\alpha^2 (u_\alpha - r^{-2} \sum_\beta x_\beta^2 u_\beta)$, has been studied extensively in the context of linear programming [8, 11, 20] and convex analysis [4]. On the other hand, the game-theoretic properties of the first order log-barrier system have not been investigated, so it is not easy to make a comparison with the benchmark case of the replicator dynamics.⁶ As we shall see, (I-RD) and (I-LD) will turn out to behave in qualitatively different ways, so the geometry of the simplex plays a crucial role in determining the long-term properties of the resulting inertial dynamics.

3.3. Friction and Dissipation of Energy. By the definition of a conservative force, it follows that the total energy

$$E(x, v) = \frac{1}{2} \langle v, v \rangle + V(x) \quad (3.22)$$

of the system will remain constant along the solutions of (ID) when $u_\alpha = -\frac{\partial V}{\partial x_\alpha}$ for some potential function $V \in C^\infty(X^\circ)$ and $\eta = 0$ (see e.g. [19] for a proof of this fact in a general Riemannian setting). On the other hand, for $\eta > 0$, we have the following dissipation result:

⁵One might be tempted to interpret the factor $1/2$ in (I-RD) as a change of time with respect to (2-RD), but the presence of \dot{x}^2 precludes as much.

⁶For a discussion on this topic, see the forthcoming paper [22].

Proposition 3.2. *Let $x(t)$ be a solution orbit of (ID). Then, the total energy $E = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle + V(x)$ of the system will be non-increasing along $x(t)$; in particular, we will have:*

$$\dot{E} = -2\eta K = -\eta \|\dot{x}\|^2, \quad (3.23)$$

where $K = \frac{1}{2}\|\dot{x}\|^2$ is the particle's kinetic energy.

Proof. By differentiating (3.22), we readily obtain:

$$\begin{aligned} \dot{E} &= \nabla_{\dot{x}} E = \frac{1}{2} \nabla_{\dot{x}} \langle \dot{x}, \dot{x} \rangle + \nabla_{\dot{x}} V = \langle \nabla_{\dot{x}} \dot{x}, \dot{x} \rangle + \langle dV | \dot{x} \rangle \\ &= \left\langle \frac{D^2 x}{Dt^2}, \dot{x} \right\rangle + \langle \text{grad } V, \dot{x} \rangle = \langle -\text{grad } V - \eta \dot{x}, \dot{x} \rangle + \langle \text{grad } V, \dot{x} \rangle \\ &= -\eta \langle \dot{x}, \dot{x} \rangle = -2\eta K, \end{aligned} \quad (3.24)$$

where we used the metric compatibility (2.15) of ∇ in the first line, and the definition of the dynamics (ID) in the second. \square

This proposition essentially shows that, for $\eta > 0$, the system's total energy E is a Lyapunov function for (ID). This is the main difference between (ID) and the Hessian–Riemannian gradient flows studied in [4]: in the first order regime, it is the potential V that plays the role of a Lyapunov function, and not the system's mechanical energy $E = K + V$. To recover the first order context, it suffices to include the mass m of the particle in our analysis: by replacing $D^2 x / Dt^2$ by $m D^2 x / Dt^2$ in (HBF*) and K by mK in (3.22), the first order dynamics of [4] are obtained in the dissipative, “light ball” limit $m \rightarrow 0$.

3.4. Euclidean coordinates. In our derivation of the dynamics (ID), we endowed X with a Riemannian structure and then expressed Newton's second law in the corresponding covariant form. The advantage of this approach is that the configuration space of the dynamics retains its original form; on the downside however, the geometry of X° is far from Euclidean, so intuition may be led astray and explicit calculations are made harder by the coordinate expressions for g . An alternative approach (which turns out to be very convenient from a calculational standpoint) is to isometrically embed X° in some ambient Euclidean space and then write the ordinary (Euclidean) version of Newton's second law for a particle constrained to move on this embedding.

Formally, recalling that X° inherits its Riemannian structure from $(\mathbb{R}_{++}^{n+1}, g)$, the above boils down to finding a *Euclidean embedding* $\Phi: \mathbb{R}_{++}^{n+1} \rightarrow \mathbb{R}^m$, i.e. an injective immersion of X° such that

$$g(E_\alpha, E_\beta) = \delta(\Phi_* E_\alpha, \Phi_* E_\beta) \quad \text{for all } \alpha, \beta = 0, 1, \dots, n, \quad (3.25)$$

where δ denotes the standard Euclidean metric of \mathbb{R}^m and $\Phi_* E_\alpha$ is the push-forward of E_α under Φ , viz.

$$\Phi_* E_\alpha = \sum_\beta \frac{\partial \Phi_\beta}{\partial x_\alpha} \bar{E}_\beta, \quad (3.26)$$

with $\{\bar{E}_\beta\}$ denoting the standard basis of the target space \mathbb{R}^m .

Remark 3.4. In what follows, vectors taken in a Euclidean space will be distinguished by an overline (such as \bar{E}_α above); also, we will use “ \cdot ” to denote the ordinary Euclidean inner product and $|\cdot|$ for the Euclidean norm.

That such an embedding exists is a consequence of the celebrated Nash–Kuiper embedding theorem [16, 24]; however, the Nash–Kuiper theorem does not provide an explicit construction method in the general case. Fortunately, in our Hessian–Riemannian framework, the situation is considerably simpler: indeed, let $\phi: (0, +\infty) \rightarrow \mathbb{R}$ be smooth and increasing, and consider the embedding $\Phi: \mathbb{R}_{++}^{n+1} \rightarrow \mathbb{R}^{n+1}$ with $(x_0, \dots, x_n) \mapsto (\phi(x_0), \dots, \phi(x_n))$. Then, if we denote the standard coordinates of the target space \mathbb{R}^{n+1} by (ξ_0, \dots, ξ_n) and its corresponding basis by $\{\bar{E}_\alpha\}_{\alpha=0}^n$ as above, we will have $\Phi_* E_\alpha = \sum_\beta \frac{\partial \xi_\beta}{\partial x_\alpha} \bar{E}_\beta = \phi'(x_\alpha) \bar{E}_\alpha$. Therefore, for Φ to be an isometry, (3.25) shows that it suffices to have

$$g_{\alpha\beta} = \delta(\phi'(x_\alpha) \bar{E}_\alpha, \phi'(x_\beta) \bar{E}_\beta) = \delta_{\alpha\beta} \phi'(x_\alpha) \phi'(x_\beta), \quad (3.27)$$

or, more concisely:

$$\phi'(x) = \sqrt{\theta''(x)}. \quad (3.28)$$

Motivated by the above, we will say that the variables $\xi = \Phi(x)$ with

$$\xi_\alpha = \phi(x_\alpha) \quad (\alpha = 0, 1, \dots, n) \quad (\text{EC})$$

are *Euclidean coordinates* for $(\mathbb{R}_{++}^{n+1}, g)$ if $\phi: (0, +\infty) \rightarrow \mathbb{R}$ satisfies (3.28) for all $x > 0$. Correspondingly, with $\dot{\xi}_\alpha = \phi'(x_\alpha) \dot{x}_\alpha = \sqrt{\theta''_\alpha} \dot{x}_\alpha$ and $\ddot{\xi}_\alpha = \theta''_\alpha / (2\sqrt{\theta''_\alpha}) \dot{x}_\alpha^2 + \sqrt{\theta''_\alpha} \ddot{x}_\alpha$, some algebra yields the following expression for the inertial dynamics (ID) in Euclidean coordinates:

$$\ddot{\xi}_\alpha = \frac{1}{\sqrt{\theta''_\alpha}} \left(u_\alpha - \sum_\beta \left(\Theta''_h / \theta''_\beta \right) u_\beta \right) + \frac{1}{2} \frac{1}{\sqrt{\theta''_\alpha}} \sum_\beta \Theta''_h \theta''_\beta / (\theta''_\beta)^2 \dot{\xi}_\beta^2 - \eta \dot{\xi}_\alpha. \quad (\text{E-ID})$$

It is then interesting to note the following with regards to (E-ID):

Remark 3.5. In the case of a conservative force with potential V , we will have $\frac{\partial V}{\partial \xi_\alpha} = \frac{\partial V}{\partial x_\alpha} \frac{dx_\alpha}{d\xi_\alpha} = -u_\alpha / \sqrt{\theta''_\alpha}$, so the first term of (E-ID) may simply be interpreted as (minus) the Euclidean gradient of V w.r.t. ξ projected on the system's *configuration manifold*:

$$S = \Phi(X^\circ) = \{\xi \in \mathbb{R}^{n+1} : \sum_\beta \phi^{-1}(\xi_\beta) = 1\}. \quad (3.29)$$

Indeed, the (Euclidean) gradient $N_S = \overline{\text{grad}} \left(\sum_\beta \phi^{-1}(\xi_\beta) \right) = \sum_\beta (\theta''_\beta)^{-1/2} \bar{E}_\beta$ of the configuration constraint $\sum_\beta \phi^{-1}(\xi_\beta) = 1$ will be normal to S by definition, so the projection of $\overline{\text{grad}} V = -\sum_\beta (\theta''_\beta)^{-1/2} u_\beta \bar{E}_\beta$ on S will be:

$$\begin{aligned} \text{pr}_S \overline{\text{grad}} V &= \overline{\text{grad}} V - \text{pr}_{N_S} \overline{\text{grad}} V \\ &= -\sum_\alpha (\theta''_\alpha)^{-1/2} u_\alpha \bar{E}_\alpha + \Theta''_h \sum_\beta u_\beta / \theta''_\beta \sum_\alpha (\theta''_\alpha)^{-1/2} \bar{E}_\alpha, \end{aligned} \quad (3.30)$$

which is simply the first term of (E-ID). In other words, the first term of (E-ID) represents the *tangential force* which drives the system along the (projection of the) gradient of V on the configuration manifold S .

Remark 3.6. Orthogonally to the above, the second term of (E-ID) represents the *centripetal force* which constrains the dynamics to remain on the configuration manifold S and which is thus normal to S [6, 19]. To see this, note that if $\xi(t) \in S$, we will have $\sum_\beta \phi^{-1}(\xi_\beta) = 1$, and hence:

$$\frac{d}{dt} \sum_\alpha \phi^{-1}(\xi_\alpha(t)) = \sum_\alpha \dot{\xi}_\alpha / \phi'(x_\alpha) = \sum_\alpha \dot{\xi}_\alpha / \sqrt{\theta''_\alpha} = 0. \quad (3.31)$$

Thus, given that the components of the centripetal force are proportional to $(\theta''_\alpha)^{-1/2}$, it follows that the force itself will be normal to $\dot{\xi}$, and hence to S . In classical mechanics, this fundamental property of holonomically constrained systems (i.e. systems constrained to move on a submanifold of \mathbb{R}^{n+1}) is known as *d'Alembert's principle of virtual displacements* [6, 19]. Essentially, this principle implies that centripetal forces produce no work along the orbits of the system, a property which will be extremely important to us in Section 6.

Example 5. In the case of the Shahshahani metric, (3.28) becomes $\phi'(x) = 1/\sqrt{x}$, so we readily obtain the Euclidean coordinates $\xi_\alpha = 2\sqrt{x_\alpha}$; this change of variables was first considered by Akin [1] and has become known in the literature as *Akin's transformation* [27]. Importantly, the image of X° in this embedding is just an open orthant of an n -dimensional sphere of radius 2 (simply note that $\xi_\beta^2 = 4x_\beta$, so $\sum_\beta \xi_\beta^2 = 4$), so the Shahshahani metric on the simplex is in all ways equivalent to the standard round metric on the sphere. Accordingly, the inertial replicator dynamics (I-RD) may be expressed in Euclidean coordinates as:

$$\ddot{\xi}_\alpha = \frac{1}{2}\dot{\xi}_\alpha \left(u_\alpha - \frac{1}{4} \sum_\beta \xi_\beta^2 u_\beta \right) - \frac{1}{2}\dot{\xi}_\alpha K - \eta \dot{\xi}_\alpha, \quad (\text{E-RD})$$

where $K = \frac{1}{2} \sum_\beta \dot{\xi}_\beta^2$ represents the system's kinetic energy.

Example 6. In the case of the log-barrier metric, we will have $\phi'(x) = 1/x$, so the metric's Euclidean coordinates will be $\xi_\alpha = \phi(x_\alpha) = \log x_\alpha$. Under this transformation, X° will be mapped isometrically to the hypersurface $S = \{\xi \in \mathbb{R}^{n+1} : \xi_\alpha < 0 \text{ and } \sum_\beta e^{\xi_\beta} = 1\}$ which is now a closed hypersurface of \mathbb{R}^{n+1} that is unbounded in all directions – in stark contrast to the Shahshahani case above. In these transformed variables, the log-barrier dynamics (I-LD) then become:

$$\ddot{\xi}_\alpha = e^{\xi_\alpha} \left(u_\alpha - r^{-2} \sum_\beta e^{2\xi_\beta} u_\beta \right) - r^{-2} e^{\xi_\alpha} \sum_\beta e^{\xi_\beta} \dot{\xi}_\beta^2 - \eta \dot{\xi}_\alpha, \quad (\text{E-LD})$$

where $r^2 = \sum_\beta x_\beta^2 = \sum_\beta e^{2\xi_\beta}$.

We thus deduce an important qualitative difference between the dynamics (I-RD) and (I-LD) which is not easy to see when looking at the dynamics on the simplex: the inertial replicator dynamics (I-RD) correspond to ordinary Newtonian motion on a bounded portion of a sphere, whereas the inertial log-barrier dynamics (I-LD) correspond to motion constrained on a closed and unbounded hypersurface of \mathbb{R}^{n+1} . As we shall see in Section 6, this difference will be the deciding factor in whether solutions of (ID) exist for all time or not.

4. LONG-TERM OPTIMIZATION PROPERTIES

As a motivating precursor to the game-theoretic analysis of Section 5, we will examine here the convergence properties of the inertial dynamics (ID) in optimization problems defined over polytopes. In this context, the problem's objective function will correspond to the potential function of the inertial dynamics (ID), and the central question that we seek to answer is whether the local minimizers of the potential are attracting or not.

To that end, recall first that the problem of minimizing a function $V_0: P \rightarrow \mathbb{R}$ over a polytope P with $n+1$ vertices may be pulled back to the unit n -dimensional simplex $X \equiv \Delta(n+1) \subseteq \mathbb{R}^{n+1}$ simply by considering a linear map

which takes the vertices of P to those of X in a bijective fashion. As a result, we will focus here on the (smooth) optimization problem:

$$\begin{aligned} & \text{minimize} && V(x) \\ & \text{subject to} && x_\beta \geq 0, \sum_{\beta=0}^n x_\beta = 1, \end{aligned} \tag{P}$$

where $V: X \rightarrow \mathbb{R}$ is a C^∞ -smooth objective function.^{7,8}

Physical intuition and the principle of energy minimization suggest that (ID) will indeed be attracted to local minimizers of V ; in what follows, we will show that this is indeed the case when the system starts with low energy near a proper local minimizer of V (i.e. a critical point of V with $\text{Hess}_q(V) \succ 0$ on $T_q X$). However, given that we are interested in the long-term convergence properties of (ID), our analysis will rest on the well-posedness property:

The solution orbits $x(t)$ of the dynamics exist for all time. (WP)

We will address the issue of whether this property holds or not in Section 6; for the moment however, it will be much more convenient to retain (WP) as a working hypothesis and to condition our presentation on it. With this in mind, our first result is the following:

Proposition 4.1. *Let $x(t)$ be a solution trajectory of (ID) which is defined for all $t \geq 0$. If $\eta > 0$, then $\lim_{t \rightarrow \infty} \dot{x}(t) = 0$.*

Before proving Proposition 4.1, it will be helpful to establish the following intermediate result which is of independent interest:

Lemma 4.2. *Let $x(t)$ be a solution trajectory of (ID) which is defined for all $t \geq 0$. Then, the rate of change of the system's kinetic energy is bounded from above for all $t \geq 0$.*

Proof. By differentiating K with respect to time, we readily obtain:

$$\begin{aligned} \dot{K} &= \nabla_{\dot{x}} K = \frac{1}{2} \nabla_{\dot{x}} \langle \dot{x}, \dot{x} \rangle = \langle \nabla_{\dot{x}} \dot{x}, \dot{x} \rangle = \langle -\text{grad } V - \eta \dot{x}, \dot{x} \rangle \\ &= -\langle dV|_{\dot{x}} \rangle - \eta \|\dot{x}\|^2 = -\sum_{\beta} \frac{\partial V}{\partial x_{\beta}} \dot{x}_{\beta} - \eta \sum_{\beta} \theta''_{\beta} \dot{x}_{\beta}^2 \\ &\leq A \sum_{\beta} |\dot{x}_{\beta}| - \eta B \sum_{\beta} \dot{x}_{\beta}^2, \end{aligned} \tag{4.1}$$

where $A = \sup \left| \frac{\partial V}{\partial x_{\beta}} \right|$ and $B = \inf \{ \theta''(x) : x \in (0, 1) \}$. With A finite and $B > 0$ (on account of the fact that $V \in C^\infty(X)$ and the Legendre properties of θ respectively), the maximum value of the above expression will be $(n+1)A^2/(4\eta B)$, so \dot{K} is bounded from above. \square

Proof of Proposition 4.1. Let $E(t) = \frac{1}{2} \|\dot{x}(t)\|^2 + V(x(t))$ be the system's energy at time t . Then, by Proposition 3.2, we will have $\dot{E} = -\eta \|\dot{x}\|^2 = -2\eta K \leq 0$, which shows that $E(t)$ will decrease to some value $E^* \in \mathbb{R}$. This also shows that K is

⁷In what follows, it will be convenient to assume that V is actually the restriction of a smooth function defined on an open neighborhood of X in \mathbb{R}^{n+1} . To do away with this technicality, one simply needs to view X as a manifold with boundary and represent V in local coordinates, but this would needlessly complicate the issue. We are also assuming C^∞ smoothness to remain in the category of smooth manifolds; our results continue to hold in the C^2 category.

⁸Of course, by pulling back V_0 to X in this way, the dimensionality of the problem might explode. To counter this issue, a better approach would be to derive the inertial dynamics (ID) directly on P_0 , but since these calculations would take us too far afield, we will postpone them for the future.

integrable with $\int_0^\infty K(s) ds = \frac{1}{2\eta} (E(0) - E^*) < \infty$. This is a strong indication that $\lim_{t \rightarrow \infty} K(t) = 0$, but since there exist positive integrable functions which do not converge to 0 as $t \rightarrow \infty$, our assertion does not yet follow.

Assume thus that $\limsup_{t \rightarrow \infty} K(t) = 3\varepsilon > 0$. In that case, there exists by continuity an increasing sequence of times $t_n \rightarrow \infty$ such that $K(t_n) > 2\varepsilon$ for all n . Accordingly, let $s_n = \sup\{t : t \leq t_n \text{ and } K(t) < \varepsilon\}$: since K is integrable and non-negative, we will have $s_n \rightarrow \infty$ (because $\liminf K(t) = 0$), so, by descending to a subsequence of t_n if necessary, we may assume without loss of generality that $s_{n+1} > t_n$ for all n . Hence, if we let $J_n = [s_n, t_n]$, we will have:

$$\int_0^\infty K(s) ds \geq \sum_{n=1}^\infty \int_{J_n} K \geq \varepsilon \sum_{n=1}^\infty |J_n|, \quad (4.2)$$

which shows that the Lebesgue measure $|J_n|$ of J_n vanishes as $t \rightarrow \infty$. Consequently, by the mean value theorem, it follows that there exists $\xi_n \in (s_n, t_n)$ such that

$$\dot{K}(\xi_n) = \frac{K(t_n) - K(s_n)}{|J_n|} > \frac{\varepsilon}{|J_n|}, \quad (4.3)$$

and since $|J_n| \rightarrow 0$, we conclude that $\limsup \dot{K}(t) = \infty$. However, this contradicts the boundedness of $\dot{K}(t)$ established by Lemma 4.2, so $K(t) \rightarrow 0$, and hence $\dot{x}(t) \rightarrow 0$ as claimed. \square

Remark 4.1. The proof technique above may easily be adapted to the Euclidean case and thus leads to an alternative proof of the velocity integrability and convergence part of Theorem 2.1 in [2]; furthermore, even if we add a Hessian-driven damping term in (ID) as in [3], the estimate (4.1) remains essentially unchanged and our approach may also be used to prove the corresponding claim of Theorem 2.1 of [3] as well.

Proposition 4.1 shows that the solution orbits of (ID) slow down, but this does not mean that they converge; for instance, they could approach a non-singleton invariant set of (ID) with vanishing velocity. That said, if an orbit spends an arbitrarily long amount of time in the vicinity of some point $q \in X$, then q must be a critical point of V restricted to the subface spanned by $\text{supp}(q)$:

Proposition 4.3. *Let $x(t)$ be a solution orbit of (ID) that is defined for all $t \geq 0$. Assume further that for every $\delta > 0$ and for every $T > 0$, there exists an interval J of length at least T such that $\max_\alpha \{|x_\alpha(t) - q_\alpha|\} < \delta$ for all $t \in J$. Then:*

$$\left. \frac{\partial V}{\partial x_\alpha} \right|_q = \left. \frac{\partial V}{\partial x_\beta} \right|_q \quad \text{for all } \alpha, \beta \in \text{supp}(q). \quad (4.4)$$

For the proof of this proposition, we will need the following preparatory lemma:

Lemma 4.4. *Let $\xi: [a, b] \rightarrow \mathbb{R}$ be a smooth curve in \mathbb{R} such that*

$$\ddot{\xi} + \eta \dot{\xi} \leq -m, \quad (4.5)$$

for some $\eta \geq 0$, $m > 0$ and for all $t \in [a, b]$. Then, for all $t \in [a, b]$, we will have:

$$\xi(t) \leq \xi(a) + \begin{cases} \eta^{-1} (\dot{\xi}(a) + m\eta^{-1}) (1 - e^{-\eta(t-a)}) - m\eta^{-1}(t-a) & \text{if } \eta > 0, \\ \dot{\xi}(a)(t-a) - \frac{1}{2}m(t-a)^2 & \text{if } \eta = 0. \end{cases} \quad (4.6)$$

Proof. The case $\eta = 0$ is trivial to dispatch simply by integrating (4.5) twice. On the other hand, for $\eta > 0$, if we multiply both sides of (4.5) with $\exp(\eta t)$ and integrate, then we obtain

$$\dot{\xi}(t) \leq \dot{\xi}(a)e^{-\eta(t-a)} - m\eta^{-1} \left(1 - e^{-\eta(t-a)}\right), \quad (4.7)$$

and our assertion follows by integrating a second time. \square

Proof of Proposition 4.3. Set $u_\beta = -\frac{\partial V}{\partial x_\beta}$, $\beta = 0, \dots, n$, and let α be such that $u_\alpha(q) \leq u_\beta(q)$ for all $\beta \in \text{supp}(q)$; furthermore, assume that $u_\alpha(q) \neq u_\gamma(q)$ for some $\gamma \in \text{supp}(q)$. We will then have $u_\alpha(q) - \sum_{\beta \in \text{supp}(q)} (\Theta_h''(q)/\theta_\beta''(q)) u_\beta(q) < -m' < 0$ for some $m' > 0$, and hence, by continuity, we may find some $m > 0$ such that

$$\theta_\alpha''(x)^{-1/2} \left(u_\alpha(x) - \sum_{\beta} \left(\Theta_h''(x)/\theta_\beta''(x) \right) u_\beta(x) \right) < m < 0 \quad (4.8)$$

for all $x \in U_\delta \equiv \{x : \max_\beta |x_\beta - q_\beta| < \delta\}$ and for all sufficiently small $\delta > 0$ (simply recall that $\lim_{x \rightarrow 0^+} \theta''(x) = +\infty$ and that $\theta_\alpha''(q) > 0$).

That being so, fix $\delta > 0$ as above, and let $M > 0$ be such that $x_\alpha - q_\alpha < -\delta$ whenever the Euclidean coordinates $\xi_\alpha = \phi(x_\alpha)$ of (EC) satisfy $\xi_\alpha < -M$. Choose also some sufficiently large $T > 0$; then, by assumption, there exists an interval $J = [a, b]$ with length $b - a \geq T$ and such that $x(t) \in U_\delta$ for all $t \in J$. Since $\lim_{t \rightarrow \infty} \dot{x}_\alpha(t) = 0$ by Proposition 4.1, we may also assume that the interval $J = [a, b]$ is such that $\dot{\xi}_\alpha(a)$ is itself sufficiently small (simply note that if x_α is bounded away from 0, $\dot{\xi}_\alpha = \phi'(x_\alpha)\dot{x}_\alpha$ cannot become arbitrarily large).

In this manner, the Euclidean presentation (E-ID) of the dynamics readily yields

$$\ddot{\xi}_\alpha \leq -m + \frac{1}{2} \frac{1}{\sqrt{\theta_\alpha''}} \sum_{\beta} \Theta_h'' \theta_\beta''' / (\theta_\beta'')^2 \dot{\xi}_\beta^2 - \eta \dot{\xi}_\alpha < -m - \eta \dot{\xi}_\alpha \quad \text{for all } t \in J, \quad (4.9)$$

where the second inequality follows from the regularity assumption $\theta'''(x) < 0$. However, with T large enough and $\dot{\xi}_\alpha(a)$ small enough, Lemma 4.4 shows that we will have $\xi_\alpha(t) < -M$ for large enough $t \in J$, and hence $x(t) \notin U_\delta$, a contradiction. We thus conclude that $u_\alpha(q) = u_\gamma(q)$ for all $\alpha, \gamma \in \text{supp}(q)$, as claimed. \square

Proposition 4.3 shows that if $x(t)$ converges to some $q \in X$, then q must be a restricted critical point of V in the sense of (4.4). More generally, the following lemma establishes that any ω -limit of (ID) has this property:

Lemma 4.5. *Let x^* be an ω -limit of (ID), and let U be a neighborhood of x^* in X . Then, for every $T > 0$, there exists an interval J of length at least T such that $x(t) \in U$ for all $t \in J$.*

Proof. Fix a neighborhood U of x^* in X , and let $U_\delta = \{x : \max_\beta |x_\beta - x_\beta^*| < \delta\}$ be a δ -neighborhood of x^* in the L^∞ norm of \mathbb{R}^{n+1} with $U_\delta \cap X \subseteq U$. By assumption, there exists an increasing sequence of times $t_n \rightarrow \infty$ such that $x(t_n) \rightarrow x^*$, so we can take $x(t_n) \in U_{\delta/2}$ for all n . Moreover, let $t'_n = \inf\{t : t \geq t_n \text{ and } x(t) \notin U_\delta\}$ be the first exit time of $x(t)$ from U_δ after t_n , and assume ad absurdum that $t'_n - t_n < M$ for some $M > 0$ and for all n . Then, by descending to a subsequence

of t_n if necessary, we will have $|x_\alpha(t'_n) - x_\alpha(t_n)| > \delta/2$ for some α and for all n . Hence, by the mean value theorem, we obtain some $\tau_n \in (t_n, t'_n)$ such that

$$|\dot{x}_\alpha(\tau_n)| = \frac{|x_\alpha(t'_n) - x_\alpha(t_n)|}{t'_n - t_n} > \frac{\delta}{2M} \quad \text{for all } n, \quad (4.10)$$

implying in particular that $\limsup |\dot{x}_\alpha(t)| > \delta/(2M) > 0$ in contradiction to Proposition 4.1. We thus conclude that the difference $t'_n - t_n$ is unbounded, i.e. for every $\delta > 0$ and for every $T > 0$, there exists an interval J of length at least T such that $x(t) \in U_\delta$ for all $t \in J$. \square

Even though the above properties of (ID) are interesting in themselves (e.g. compare with the conclusions of Theorem 5.2 for games), for now they will mostly serve as stepping stones to the following asymptotic convergence result:

Theorem 4.6. *Assume that the inertial dynamics (ID), $\eta > 0$, are well-posed in the sense of (WP), and let $q \in X$ be a local minimizer of V with $\text{Hess}_q V \succ 0$ on $T_q X$. Then, if $x(t)$ starts close enough to q and with sufficiently low energy, we will have $\lim_{t \rightarrow \infty} x(t) = q$.*

Proof. Let x^* be an ω -limit of $x(t)$. By Lemma 4.5, $x(t)$ will be spending arbitrarily long time intervals near x^* , so Proposition 4.3 shows that x^* will satisfy the stationarity condition (4.4), viz. $\partial_\alpha V|_{x^*} = \partial_\beta V|_{x^*} = -u^*$ for all $\alpha, \beta \in \text{supp}(x^*)$. By the theorem's assumptions, this will allow then us to conclude that q is the unique ω -limit of $x(t)$, i.e. $\lim_{t \rightarrow \infty} x(t) = q$.

Indeed, if $x(t)$ starts close enough to q and with sufficiently low energy, Proposition 3.2 shows that every ω -limit of $x(t)$ must also lie close enough to q (simply note that $V(x(t))$ can never exceed the initial energy $E(0)$ of $x(t)$); as a result, the support of any ω -limit of $x(t)$ will contain that of q . However, with $\text{Hess}_q V \succ 0$, V will be strongly convex near q , and since x^* itself lies close enough to q , we will have

$$\sum_\beta \frac{\partial V}{\partial x_\beta} \Big|_{x^*} (q_\beta - x_\beta^*) \leq V(q) - V(x^*) \leq 0, \quad (4.11)$$

with equality if and only if $x^* = q$. On the other hand, with $\text{supp}(q) \subseteq \text{supp}(x^*)$ we also get

$$\sum_{\beta=0}^n \frac{\partial V}{\partial x_\beta} \Big|_{x^*} (q_\beta - x_\beta^*) = \sum_{\beta \in \text{supp}(x^*)} \frac{\partial V}{\partial x_\beta} \Big|_{x^*} (q_\beta - x_\beta^*) = -u^* \sum_{\beta \in \text{supp}(x^*)} (q_\beta - x_\beta^*) = 0, \quad (4.12)$$

so $x^* = q$, and our proof is complete. \square

Remark 4.2. Since the total energy $E(t)$ of the system is decreasing, Theorem 4.6 implies that whenever $x(t)$ starts close to q with low energy, then $x(t)$ will stay close to q and eventually converge to it. This formulation is almost equivalent to q being asymptotically stable in (ID), and if q is itself interior, the two statements are indeed equivalent. For $q \in \text{bd}(X)$ however, asymptotic stability is more cumbersome to define because the structure of the phase space of the dynamics (ID) changes at every subface of X , so we opted to stay with the simpler formulation of Theorem 4.6.

Remark 4.3. We should also note here that the positive-definiteness requirement $\langle z | \text{Hess}_q(V) | z \rangle > 0$ for every nonzero admissible tangent vector $z \in T_x X$ can be relaxed: all we need is that there be no other point x^* near q such that $\partial_\alpha V|_{x^*} = \partial_\beta V|_{x^*}$ for all $\alpha, \beta \in \text{supp}(x^*)$.

Remark 4.4. Theorem 4.6 is a local convergence result and does not exploit global properties of the objective function (such as convexity or geodesic convexity) in order to establish global convergence. Even though physical intuition suggests that this should be easily possible, the mathematical analysis is quite convoluted due to the boundary behavior of the covariant correction term of (ID) (i.e. the centripetal force of (E-ID) which constrains the dynamics (E-ID) to S). Since the main focus of this paper is the rationality properties of the game-theoretic extension of (ID), we will not address this point here and delegate the global analysis of (ID) to future work.

5. RATIONALITY ANALYSIS AND LONG-TERM BEHAVIOR IN GAMES

In this section, we reconnect with our original objective which was to adapt the “heavy ball with friction” dynamics to a game-theoretic setting, and to study the rationality properties of the resulting dynamics.

5.1. Inertial game dynamics. As we mentioned earlier, a foundational interpretation of the replicator dynamics is that each player looks at the direction of individual steepest ascent of his payoff function with respect to the so-called Shahshahani metric on the simplex, and then tracks this direction in the hopes of maximizing his personal gains [15, 28]. Building on this interpretation, our aim in this section will be to derive a class of inertial learning dynamics based on the similarly simple premise that the unilateral gradient of each player’s payoff function acts as a *force* on the player’s choices and thus determines the acceleration of the player’s learning trajectory (instead of its velocity).

Formally, let $\mathfrak{G} \equiv \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$ be a game in normal form and let $X \equiv \prod_k \Delta(\mathcal{A}_k)$ be the game’s mixed strategy space. In view of (3.20), if θ_k is a player-specific Legendre-type kernel satisfying (L), the *unilateral gradient* of each player’s payoff function $u_k: X \rightarrow \mathbb{R}$ is defined as:

$$\text{grad}_k u_k = \sum_\alpha \frac{1}{\theta''_{k\alpha}} \left(u_{k\alpha} - \sum_\beta \left(\Theta''_{k,h} / \theta''_{k\beta} \right) u_{k\beta} \right) E_{k\alpha}, \quad (5.1)$$

where, as before, $\theta''_{k\alpha}$ is shorthand for $\theta''_k(x_{k\alpha})$, $\Theta''_{k,h}$ denotes the harmonic aggregate $\Theta''_{k,h} = \left(\sum_\beta 1 / \theta''_{k\beta} \right)^{-1}$, and we have used the fact that

$$\frac{\partial u_k}{\partial x_{k\alpha}} = u_k(\alpha; x_{-k}) = u_{k\alpha}(x). \quad (5.2)$$

In other words, the unilateral gradient $\text{grad}_k u_k$ of u_k is simply the gradient of u_k with respect to the Hessian–Riemannian structure generated by θ_k on $X_k \equiv \Delta(\mathcal{A}_k)$, and evaluated with the strategies of k ’s opponents kept fixed for the calculation.

In this way, the *inertial game dynamics* that we will focus on the rest of the paper will be:

$$\frac{D^2 x_k}{Dt^2} = \text{grad}_k u_k - \eta_k \dot{x}_k, \quad (5.3)$$

or, in components:

$$\begin{aligned}\ddot{x}_{k\alpha} = & \frac{1}{\theta''_{k\alpha}} \left(u_{k\alpha} - \sum_{\beta}^k \left(\Theta''_{k,h}/\theta''_{k\beta} \right) u_{k\beta} \right) \\ & - \frac{1}{2} \frac{1}{\theta''_{k\alpha}} \left(\theta'''_{k\alpha} \dot{x}_{k\alpha}^2 - \sum_{\beta}^k \left(\Theta''_{k,h}/\theta''_{k\beta} \right) \theta'''_{k\beta} \dot{x}_{k\beta}^2 \right) - \eta_k \dot{x}_{k\alpha},\end{aligned}\quad (\text{IGD})$$

where $\eta_k \geq 0$ is a player-specific friction coefficient. Needless to say, in the case of the Shahshahani kernel $\theta_S(x) = x \log x$ and the logarithmic barrier $\theta_L(x) = -\log x$, we obtain respectively the inertial replicator dynamics (I-RD) and the inertial log-barrier dynamics (I-LD), the only difference now being that every quantity therein is player-specific.

Remark 5.1. The dynamics (IGD) are only defined in the interior X° of the game's strategy space X , and in view of the blow-up of θ''' near 0, it is not immediately apparent how these dynamics may be extended to all of X . When we need such an extension, (IGD) will be extended to the (relative) interior of any subface X' of X simply by restricting all sums in (IGD) to be taken over the actions $\alpha \in \mathcal{A}_k$, $k \in \mathcal{N}$, that are supported in X' , and by restricting initial velocities $\dot{x}(0)$ to lie in $T_{x(0)}X'$ instead of $T_{x(0)}X$. In this way, every subface X' will be an invariant manifold of (IGD), a property which in a first order setting essentially characterizes the class of *imitative dynamics* [9] (a terminology which refers to the fact that only strategies that are already being employed may be imitated and reproduced).

5.2. Rationality analysis and long-term behavior. Our aim will now be to examine the convergence and stability properties of (IGD) with respect to the Nash equilibria of the underlying game. To that end (and to tie our analysis with that of Section 4, we first present a preliminary result for potential games, viewed here simply as a class of (non-convex) optimization problems defined over products of simplices. Without further ado, we have:

Proposition 5.1. *Let $\mathfrak{G} \equiv \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$ be a potential game with potential function V . Assume further that the inertial game dynamics (IGD) with $\eta_k \equiv \eta > 0$ satisfy the well-posedness condition (WP), and let q be an isolated minimizer of V (and, hence, a strict Nash equilibrium of \mathfrak{G}). If $x(t)$ starts close enough to q and with sufficiently low kinetic energy $K(0) = \frac{1}{2} \|\dot{x}(0)\|^2$, then $x(t)$ will stay close to q for all $t \geq 0$ and will eventually converge to q .*

Proof. In the presence of a potential function V as in (2.2), the dynamics (5.3) become $D^2x/Dt^2 = -\text{grad } V - \eta \dot{x}$, $x \in X \equiv \prod_k \Delta(\mathcal{A}_k)$, so our claim is proven just as in the case of Theorem 4.6: Propositions 4.1 and 4.3 extend effortlessly to the case where X is a product of simplices, and even without the positive-definiteness condition $\text{Hess}_q V \succ 0$ on $T_q X$, it is easy to see that there are no other stationary points of (IGD) near an isolated equilibrium of \mathfrak{G} (cf. Remark 4.3 following Theorem 4.6). As a result, any trajectory of (IGD) which starts close to an isolated equilibrium q of \mathfrak{G} and always remains in its vicinity will eventually converge to q , and since trajectories which start near q and with sufficiently low energy have this property, our claim follows. \square

Remark 5.2. Even though Proposition 5.1 is essentially a reprise of Theorem 4.6, it is still important in terms of interpretation because it allows us to view the

potential function of a game as the potential of an actual physical system. Specifically, in view of the Euclidean transformation (EC), a potential game can now simply be seen as a classical mechanical system which is constrained to move on the configuration manifold $\Phi(X)$ of $\prod_k \mathbb{R}^{A_k}$, and which evolves under the influence of the potential function $\bar{V} = V \circ \Phi^{-1}$.

On the other hand, Proposition 5.1 does not say much for general, non-potential games. In such a context, the most well-known stability and convergence result is the *folk theorem of evolutionary game theory* [15, 27] which states that the multi-population replicator equation

$$\dot{x}_{k\alpha} = x_{k\alpha} (u_{k\alpha} - u_k) \quad (1\text{-RD})$$

has the following properties:

- I. A state is stationary if and only if it is a restricted equilibrium.
- II. If an interior solution orbit converges, its limit is Nash.
- III. If a point is Lyapunov stable, then it is also Nash.
- IV. A point is asymptotically stable if and only if it is a strict equilibrium.

In the rest of this section, our goal will be to see whether a similar theorem may be obtained in the context of the inertial game dynamics (IGD). Of course, the second order playing field is fundamentally different from the standard first order one, so one cannot expect a verbatim translation of the folk theorem to our Newtonian setting: for instance, stationarity and stability cannot be stated simply in terms of the players' initial strategy assignments $x(0) \in X$ but one needs to also prescribe the strategies' initial growth rate $\dot{x}(0)$. In spite of these differences, we have:

Theorem 5.2. *Let $\mathfrak{G} \equiv \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$ be a finite game and let $q \in X$. Assume further that the inertial dynamics (IGD) for \mathfrak{G} satisfy the well-posedness condition (WP), and let $x(t)$ be a solution orbit of (IGD) for $\eta_k \geq 0$. Then:*

- I. $x(t) = q$ for all $t \geq 0$ if and only if q is a restricted equilibrium of \mathfrak{G} (i.e. $u_{k\alpha}(q) = \max\{u_{k\beta}(q) : q_{k\beta} > 0\}$ whenever $q_{k\alpha} > 0$).
- II. If $x(t)$ is interior and $\lim_{t \rightarrow \infty} x(t) = q$, then q is a restricted equilibrium of \mathfrak{G} .
- III. If every neighborhood U of q in X admits an interior orbit $x_U(t)$ such that $x_U(t) \in U$ for all $t \geq 0$, then q is a restricted equilibrium of \mathfrak{G} .
- IV. If q is a strict equilibrium of \mathfrak{G} and $x(t)$ starts close enough to q with sufficiently low speed $\|\dot{x}(0)\|$, then $x(t)$ remains close to q for all $t \geq 0$ and $\lim_{t \rightarrow \infty} x(t) = q$.

Proof. We begin with stationarity of restricted Nash equilibria. Obviously, given how the dynamics (IGD) have been extended to the subfaces of X , it suffices to consider interior stationary equilibria. Thus, if $q \in X^\circ$ is Nash, we will have $u_{k\alpha}(q) = u_{k\beta}(q)$ for all $\alpha, \beta \in \mathcal{A}_k$, and hence also $u_{k\alpha}(q) = \sum_{\beta}^k (\Theta''_{k,h} / \theta''_{k\beta}) u_{k\beta}(q)$ for all $\alpha \in \mathcal{A}_k$. Furthermore, with $\theta''_{\alpha}(q) > 0$, the velocity-dependent terms of (IGD) will also vanish if $\dot{x}_{k\alpha}(0) = 0$ for all $\alpha \in \mathcal{A}_k$, so the initial conditions $x(0) = q$, $\dot{x}(0) = 0$, imply that $x(t) = q$ for all $t \geq 0$. Conversely, if $x(t) = q$ for all time, then we will also have $\dot{x}(t) = 0$ for all $t \geq 0$, and hence $u_{k\alpha}(q) = \sum_{\beta}^k (\Theta''_{k,h} / \theta''_{k\beta}) u_{k\beta}(q)$ for all $\alpha \in \mathcal{A}_k$, i.e. q is an equilibrium of \mathfrak{G} .

For Part II of the theorem, note that if an interior trajectory $x(t)$ converges to $q \in X$, then every neighborhood U of q in X admits an interior orbit $x_U(t)$ such

that $x_U(t)$ stays in U for all $t \geq 0$, so the claim of Part II is subsumed in that of Part III.⁹ To that end, assume ad absurdum that q has the property described above without being a restricted equilibrium, i.e. there exists $\alpha \in \text{supp}(q_k)$ with $u_{k\alpha}(q) < \max_{\beta} \{u_{k\beta}(q)\}$. As in the proof of Proposition 4.3,¹⁰ let U be a small enough neighborhood of q in the L^∞ norm of X such that

$$\theta''_{k\alpha}(x)^{-1/2} \left(u_{k\alpha}(x) - \sum_{\beta}^k \left(\Theta''_{k,h}(x) / \theta''_{k\beta}(x) \right) u_{k\beta}(x) \right) < m < 0 \quad (5.4)$$

for all $x \in U$. Then, with $x(t) \in U$ for all $t \geq 0$, the Euclidean presentation (E-ID) of the inertial dynamics (IGD) readily gives

$$\ddot{\xi}_{k\alpha} \leq -m + \frac{1}{2} \frac{1}{\sqrt{\theta''_{k\alpha}}} \sum_{\beta}^k \Theta''_{k,h} \theta''_{k\beta} / (\theta''_{k\beta})^2 \dot{\xi}_{k\beta}^2 - \eta_k \dot{\xi}_{k\alpha} < -m - \eta_k \dot{\xi}_{k\alpha} \quad \text{for all } t \geq 0, \quad (5.5)$$

so, by Lemma 4.4, we obtain $\xi_{k\alpha}(t) \rightarrow -\infty$ as $t \rightarrow \infty$. However, the definition (EC) of the Euclidean coordinates $\xi_{k\alpha}$ implies that $x_{k\alpha}(t) \rightarrow 0$ if $\xi_{k\alpha}(t) \rightarrow -\infty$, and since $q_{k\alpha} > 0$ by assumption, we obtain a contradiction which establishes our original claim.

For the final part of the theorem, assume w.l.o.g. that $q = (e_{1,0}, \dots, e_{N,0})$ be a strict equilibrium of \mathfrak{G} (recall also that only vertices of X can be strict equilibria of \mathfrak{G}). We will show that if $x(t)$ starts at rest ($\dot{x}(0) = 0$) and with initial Euclidean coordinates $\xi_{k\mu}(0)$, $\mu \in \mathcal{A}_k^* \equiv \mathcal{A}_k \setminus \{0\}$, that are sufficiently close to their lowest possible value $\phi_{k,0} \equiv \inf\{\phi_k(x) : x > 0\}$,¹¹ then $x(t) \rightarrow q$ as $t \rightarrow \infty$. The proof will remain essentially unchanged (albeit more tedious to write down) if the (Euclidean) norm of the initial velocity $\dot{\xi}(0)$ of the trajectory is bounded by some sufficiently small constant $\delta > 0$, so the theorem will follow by recalling that $\|\dot{x}(0)\| = |\dot{\xi}(0)|$.

Indeed, let U be a neighborhood of q in the L^∞ norm of X such that (5.4) holds for all $x \in U$ and for all $\mu \in \mathcal{A}_k^* \equiv \mathcal{A}_k \setminus \{0\}$ substituted in place of α . Moreover, let $U' = \Phi(U)$ be the image of U in the Euclidean coordinates $\xi = \Phi(x)$ corresponding to θ , and let $\tau_U = \inf\{t : x(t) \notin U\} = \inf\{t : \xi(t) \notin U'\}$ be the first escape time of $\xi(t) = \Phi(x(t))$ from U' . Assuming $\tau_U < +\infty$ (recall that $\xi(t)$ is assumed to exist for all $t \geq 0$), we will have $x_{k\mu}(\tau_U) \geq x_{k\mu}(0)$ and hence $\xi_{k\mu}(\tau_U) \geq \xi_{k\mu}(0)$ for some $k \in N$, $\mu \in \mathcal{A}_k^*$; consequently, there exists some $\tau' \in (0, \tau_U)$ such that $\dot{\xi}_{k\mu}(\tau') \geq 0$. By the definition of U , we will also have $\ddot{\xi}_{k\mu} + \eta_k \dot{\xi}_{k\mu} < -m < 0$ for all $t \in (0, \tau_U)$, and hence, with $\dot{\xi}(0) = 0$, the bound (4.7) in the proof of Lemma 4.4 readily yields $\dot{\xi}_{k\mu}(\tau') < 0$, a contradiction.¹² We thus conclude that $\tau_U = +\infty$, so we will also have $\ddot{\xi}_{k\mu} + \eta_k \dot{\xi}_{k\mu} < -m < 0$ for all

⁹Interestingly, this would not be so if we had formulated Part III in terms of Lyapunov stability.

¹⁰Note here that Proposition 4.3 does not apply directly because the dynamics (IGD) need not be conservative.

¹¹Recall here that by the definition of the Euclidean coordinates $\xi_{k\alpha} = \phi_k(x_{k\alpha})$, this condition is equivalent to $x(t)$ starting at a small enough neighborhood of q .

¹²One simply needs to consider the escape time $\tilde{\tau}$ from a larger neighborhood \tilde{U} of q chosen so that if $|\dot{\xi}_{k\mu}(0)| < \delta$ for some sufficiently small $\delta > 0$, then the bound (4.7) guarantees the existence of a non-positive rate of change $\dot{\xi}_{k\mu}(\tau_0)$ for some $\tau_0 < \tilde{\tau}$.

$k \in \mathcal{N}$, $\mu \in \mathcal{A}_k^*$, and for all $t \geq 0$. Lemma 4.4 then gives us $\lim_{t \rightarrow \infty} \zeta_{k\mu}(t) = -\infty$, and hence $x(t) \rightarrow q$. \square

Theorem 5.2 is our main rationality result for finite games, so some remarks are in order:

Remark 5.3. First off, it will be important to make a comparison of Theorem 5.2 to the standard first order folk theorem. To that end, we may note the following:

PART I of Theorem 5.2 is equivalent to the corresponding first order statement and basically states that mechanical and restricted equilibria coincide. on the other hand, if q is a restricted equilibrium of \mathfrak{G} and $x(0) = q$ but $\dot{x}(0) \neq 0$, then $x(t)$ will clearly not remain at q .

PART II of the theorem differs from the first order case in that it does not disallow convergence to non-Nash stationary profiles. For $\eta = 0$, the reason for this behavior is that if a trajectory $x(t)$ starts close to a restricted equilibrium q with an initial velocity pointing towards q , then $x(t)$ may escape towards q if there is only a vanishingly small force pushing $x(t)$ away from q (compare with what happens in the physical world where a projectile may escape Earth's gravitational field if given a sufficient boost). That said, we have not been able to find such a counterexample for $\eta > 0$ and we conjecture that even a small amount of friction prohibits convergence to non-Nash profiles (which, as we shall see in Section 6, correspond to points at infinity in the Euclidean dynamics (E-ID) when (IGD) is well-posed).

PART III is at the same time stronger and weaker than its first order analogue. On the one hand, the existence of a single interior trajectory that stays close to q is a much less stringent requirement than Lyapunov stability; on the other hand, and for the same reasons as before, this condition does not suffice to exclude non-Nash stationary points of (IGD).

PART IV is not exactly the same as the corresponding first order statement because the dynamics (IGD) are only defined on X° , so we cannot talk about "asymptotic stability" of a strict equilibrium. Theorem 5.2 shows instead that if $x(t)$ starts close to q and with sufficiently low speed $\|\dot{x}(0)\|$ (or, equivalently, kinetic energy $K(0) = \frac{1}{2}\|\dot{x}(0)\|^2$), then $x(t)$ will remain close and eventually converge to q . Since this result remains true when restricting (IGD) to any subface X' of X containing q , this can be seen as a form of asymptotic stability of q .¹³

Finally, Theorem 5.2 does not address the converse question of whether (IGD) admits asymptotically stable states that are not strict equilibria of \mathfrak{G} . This question is intimately tied to the behavior of the corresponding first order dynamics $\dot{x}_k = \text{grad}_k u_k$, but since a thorough analysis of these dynamics is beyond the scope of this paper, we defer this question to a forthcoming paper [22].

Remark 5.4. In the case of the Shahshahani metric and the associated inertial replicator dynamics (I-RD), it is also important to compare Theorem 5.2 above to Theorem 5.1 of [17], i.e. the folk theorem for the second order replicator dynamics (2-RD). In a nutshell, for the dynamics (2-RD), the analysis of [17] shows that:

¹³This could be formalized by considering the phase space obtained by joining the phase space of (IGD) with that of every possible restriction of (IGD) to a subface X' of X , but the gain hardly seems worth the effort (see also the relevant remark following Theorem 4.6).

- I. A state is stationary in (2-RD) if and only if it is a restricted equilibrium of \mathfrak{G} .
- II. If an interior solution orbit converges in (2-RD), then its limit is Nash.
- III. If every neighborhood of a point admits an interior orbit which always stays close to the point in question, then this point is Nash.
- IV. If an orbit starts close enough to a strict equilibrium and with sufficiently low kinetic energy, then it converges to the strict equilibrium in question; conversely, only strict equilibria have this property.^{14,15}

A first important difference between the dynamics (I-RD) and (2-RD) is that the latter characterize Nash equilibria as ω -limits of interior trajectories, whereas the former do not. However, it turns out that there is a much more important difference between (I-RD) and (2-RD): Theorem 5.2 is conditional on the well-posedness condition (WP), and as we shall see in Section 6, the inertial replicator dynamics (I-RD) are *not* well-posed.

In our view, this represents an extremely surprising aspect of the dynamics (I-RD) which differ from (2-RD) only by a factor 1/2 in the correction term $x_\alpha(\dot{x}_\alpha^2/x_\alpha^2 - \sum_\beta \dot{x}_\beta^2/x_\beta^2)$. Thanks to Akin's transformation and the presentation (E-RD) of (I-RD) in Euclidean coordinates, we are able to provide a geometric explanation of this unforeseen behavior in Section 6; that said, this also shows that the analysis of higher order dynamics on X can become a subtle affair with the end result depending heavily on the geometric considerations at hand.¹⁶

6. THE WELL-POSEDNESS DICHOTOMY

A key element in the asymptotic analysis of the previous sections is the well-posedness property (WP): if the dynamics (ID)/(IGD) are well-posed, then they exhibit several compelling long-term properties; otherwise, the very notion of asymptotic analysis breaks down. In this section, we will investigate conditions under which the well-posedness condition (WP) holds and we will present two important examples that showcase this dichotomy. On the one hand, the inertial log-barrier dynamics (I-LD) are well-posed, so the analysis of the previous sections applies unchanged; on the other hand (and somewhat surprisingly), the inertial replicator dynamics (I-RD) are *not* well-posed – despite the fact that the only difference between (I-RD) and the well-behaved second order replicator equation (2-RD) is a factor of 1/2 in their centripetal force terms.

6.1. Inexistence of global solutions in the inertial replicator dynamics. We begin by showing that the geometric barrier imposed by the blow-up of the Shahshahani metric does not suffice to keep the solutions of (I-RD) in X for all time, even

¹⁴In fact, geometrical considerations are completely absent from [17], but it is easy to see that the convergence conditions of [17] essentially boil down to the low energy condition above.

¹⁵Importantly, note that in the second order replicator dynamics (2-RD), the only states that attract nearby initial conditions with low energy are the game's strict equilibria.

¹⁶Even though the dynamics (2-RD) are not derived from metric considerations, they can be derived from a non-Euclidean connection $\tilde{\nabla}$ on \mathbb{R}^{n+1} with Christoffel symbols $\tilde{\Gamma}_{\alpha\beta}^\gamma = -\delta_{\alpha\beta\gamma}/x_\alpha$ on \mathbb{R}_{++}^{n+1} , i.e. twice the Christoffel symbols of the Shahshahani connection. Even though this connection is not compatible with the Shahshahani metric, one may still interpret (2-RD) as a covariant system, but with the caveat that energy is no longer conserved.

in the simplest case of a single agent with 2 actions. Indeed, consider the single-player zero game with action set $\mathcal{A} = \{0, 1\}$ and payoffs $u_0 = u_1 = 0$. Then, letting $x \equiv x_1 = 1 - x_0$, the dynamics (I-RD) become

$$\dot{x} = \frac{1}{2}x \left(\dot{x}/x^2 - \dot{x}^2/x - \dot{x}^2/(1-x) \right) = \frac{1}{2} \frac{1-2x}{x(1-x)} \dot{x}^2, \quad (6.1)$$

so, after the Euclidean change of variables $\xi = 2\sqrt{x}$, we readily obtain the dyadic version of (E-RD) for the zero game:

$$\ddot{\xi} = -\frac{\xi}{4-\xi^2} \dot{\xi}^2. \quad (6.2)$$

To solve this equation, let $v = \dot{\xi}$ so that $\ddot{\xi} = \dot{v} = \frac{dv}{d\xi} \frac{d\xi}{dt} = v \frac{dv}{d\xi}$. By substituting in (6.2) and separating variables, we then get

$$\frac{dv}{v} = -\frac{\xi}{4-\xi^2} d\xi. \quad (6.3)$$

Letting $\xi_0 \equiv \xi(0)$ and $v_0 \equiv v(0)$, a further integration yields

$$\log \frac{v}{v_0} = \frac{1}{2} \log \frac{4-\xi^2}{4-\xi_0^2}, \quad (6.4)$$

and solving back for $\dot{\xi} = v$, we get

$$\dot{\xi} = v = \frac{v_0}{4-\xi_0^2} \sqrt{4-\xi^2}. \quad (6.5)$$

Thus, by separating variables and integrating one last time, we obtain

$$\arcsin(\xi/2) = \arcsin(\xi_0/2) + \frac{v_0 t}{\sqrt{4-\xi_0^2}}, \quad (6.6)$$

and a little algebra finally yields the harmonic oscillation:

$$\xi = \xi_0 \cos \left(\frac{v_0 t}{\sqrt{4-\xi_0^2}} \right) + \sqrt{4-\xi_0^2} \sin \left(\frac{v_0 t}{\sqrt{4-\xi_0^2}} \right). \quad (6.7)$$

We thus see that for any interior initial position $\xi_0 \in (0, 2)$ and for any initial growth rate $v_0 \neq 0$, $\xi(t)$ will become negative in finite time, so even in the case of the zero game, the solution orbits of (I-RD) may fail to exist for all time.

In retrospect, this behavior is hardly surprising: after all, the harmonic oscillation (6.7) is simply the geodesic equation on the circle $\mathbb{S}^1 = \{(\xi_0, \xi_1) : \xi_0^2 + \xi_1^2 = 4\}$ whose positive quadrant $S = \{(\xi_0, \xi_1) \in \mathbb{S}^1 : \xi_0 > 0, \xi_1 > 0\}$ is the image of X° under the change of variables $x \mapsto \xi = 2\sqrt{x}$. More generally, recalling that (E-RD) with no forcing ($u = 0$) is just the geodesic equation on the sphere, it follows that its solutions will be represented by great circles, and such trajectories may not be contained in any orthant of the sphere for all time.¹⁷

In view of the above, we see that the failure of well-posedness for the inertial replicator dynamics (I-RD) is due to a deep geometric reason: under the Euclidean change of coordinates (EC), the strategy space of the game is transformed

¹⁷Needless to say, similar conclusions may be drawn in the presence of bounded external forcing (i.e. a nonzero game \mathfrak{G}) and/or friction.

into a bounded portion of the sphere (endowed with the standard round metric), and this bounded space cannot contain geodesics for all time. On the contrary, the second order replicator dynamics (2-RD) of [17] do not fall in this geometric framework, so, despite their external similarities, it is natural that they behave in a very different fashion.

6.2. Conditions for well-posedness. In view of the above, a fundamental question that emerges is whether any Hessian–Riemannian metric can give rise to a well-posed Newtonian evolution system for games and optimization. Since all our analysis in Sections 4 and 5 is conditional on this crucial requirement, we will devote the rest of this section to answering this question.

Reviewing the failure of well-posedness in the case of (I-RD), one expects that if the Euclidean embedding (EC) of X° produces a bounded hypersurface of \mathbb{R}^{n+1} , then the inertial dynamics (ID) will not be well-posed. Conversely, if the configuration manifold $S = \Phi(X^\circ)$ of the dynamics (E-ID) is a closed (and hence unbounded in all directions) hypersurface of \mathbb{R}^{n+1} , then a particle moving on S under a bounded force will not be able to escape to infinity in finite time. Indeed, we have:

Theorem 6.1. *Let g be a Hessian–Riemannian metric on \mathbb{R}_{++}^{n+1} and let $S = \Phi(X^\circ)$ be the image of X° under the Euclidean transformation (EC). If S is a closed hypersurface of \mathbb{R}^{n+1} , then the inertial dynamics (ID) are well-posed on X° : for any $x_0 \in X^\circ$ and for any $v_0 \in \mathbb{R}^{n+1}$, there exists a unique global solution $x(t)$ of (ID) such that $x(0) = x_0$ and $\dot{x}(0) = v_0$; conversely, if S is not closed, then the dynamics (ID) may fail to be well-posed, even for $u \equiv 0$. Finally, subject to obvious changes in notation, the same conclusions also hold for the inertial game dynamics (IGD).*

Proof. As indicated by our initial discussion, we will prove this theorem for the equivalent Euclidean dynamics (E-ID); we will also only tackle the frictionless case $\eta = 0$, the case $\eta > 0$ being entirely similar. Finally, for notational convenience, the Euclidean inner product will be denoted in what follows by $X \cdot Y$ and the corresponding norm by $|\cdot|$.

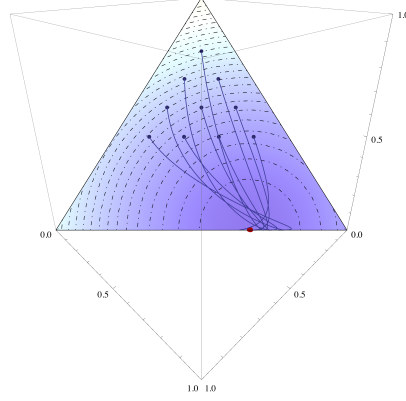
With this in mind, let $\zeta(t)$ be a local solution orbit of (E-ID) with initial conditions $\zeta(0) \equiv \zeta_0 \in S$ and $\dot{\zeta}(0) = \dot{\zeta}_0 \in T_{\zeta_0}S$; existence and uniqueness of $\zeta(t)$ follow from the classical Picard–Lindelöf theorem, so assume ad absurdum that $\zeta(t)$ only exists up to some maximal time $T > 0$. Following the physical intuition outlined above, let

$$F_\alpha = \frac{1}{\sqrt{\theta''_\alpha}} \left(u_\alpha - \sum_\beta \left(\Theta''_h / \theta''_\beta \right) u_\beta \right), \quad (6.8a)$$

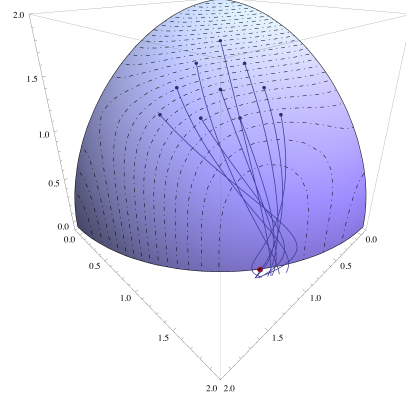
and

$$N_\alpha = \frac{\Theta''_h}{2\sqrt{\theta''_\alpha}} \sum_\beta \theta''_\beta / (\theta''_\beta)^2 \dot{\zeta}_\beta^2, \quad (6.8b)$$

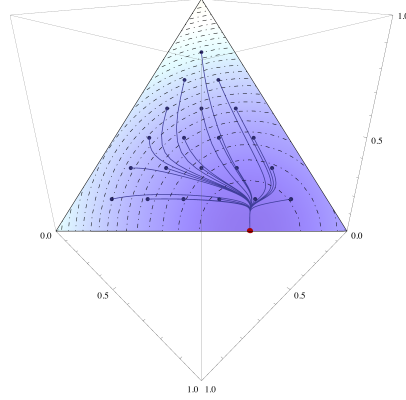
denote the tangential and centripetal forces of (E-ID) respectively. Since F is a weighted difference of bounded quantities, we will have $|F(\zeta(t))| \leq F_{\max}$ for some $F_{\max} > 0$; furthermore, as we noted in the remarks following the derivation of (E-ID), the centripetal force N will be normal to S . As a result, for all $t < T$,



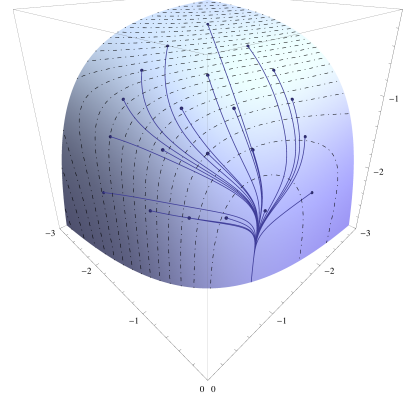
(a) The inertial replicator dynamics (I-RD).



(b) Euclidean presentation (E-RD) of (I-RD).



(c) The inertial log-barrier dynamics (I-LD).



(d) Euclidean presentation (E-LD) of (I-LD).

FIGURE 1. Long-term behavior of the inertial dynamics (ID) and their Euclidean presentation (E-ID) for the Shahshahani metric g^S (top) and the log-barrier metric g^L (bottom) – cf. (3.4) and (3.7) respectively. The surfaces to the right depict the Nash–Kuiper embedding (EC) of the simplex in \mathbb{R}^{n+1} , and the contours represent the level sets of the objective function $V: \mathbb{R}^3 \rightarrow \mathbb{R}$ with $V(x, y, z) = (x - 2/3)^2 + (y - 1/3)^2 + z^2$. As can be seen in Figs. (a) and (b), the solutions of the inertial replicator dynamics (I-RD) end up hitting the boundary of X in finite time, and thus fail to optimize V ; on the other hand, the solution orbits of (I-LD) converge globally to the minimum point of V .

the work of the resultant force $F + N$ along $\tilde{\zeta}(t)$ will be:

$$W(t) = \int_{\tilde{\zeta}} (F + N) = \int_0^t F(\tilde{\zeta}(s)) \cdot \dot{\tilde{\zeta}}(s) ds \leq F_{\max} \int_0^t |\dot{\tilde{\zeta}}(s)| ds \leq F_{\max} \ell(t), \quad (6.9)$$

where $\ell(t) = \int_0^t |\dot{\tilde{\zeta}}(s)| ds$ is the (Euclidean) length of the orbit $\tilde{\zeta}(s)$ up to time t .

On the other hand, with $F + N = \ddot{\tilde{\zeta}}$, we will also have

$$W(t) = \int_0^t \ddot{\tilde{\zeta}}(s) \cdot \dot{\tilde{\zeta}}(s) ds = \frac{1}{2} v^2(t) - \frac{1}{2} v_0^2, \quad (6.10)$$

where $v(t) = |\dot{\zeta}(t)| = \dot{\ell}(t)$ is the speed of the trajectory at time t and $v_0 \equiv |\dot{\zeta}_0|$. Combining with (6.9), we thus get the differential inequality

$$v(t) = \dot{\ell}(t) \leq \sqrt{v_0^2 + 2F_{\max}\ell(t)}, \quad (6.11)$$

which, after separating variables and integrating, gives:

$$\sqrt{v_0^2 + 2F_{\max}\ell(t)} - v_0 \leq F_{\max}t. \quad (6.12)$$

It thus follows that the speed $v(t)$ of the trajectory will be bounded by $|\dot{\zeta}(t)| = v(t) \leq v_0 + F_{\max}t$; similarly, for the total distance travelled by $\zeta(t)$, we will have $\ell(t) \leq v_0t + \frac{1}{2}F_{\max}t^2$, so $|\zeta|$ and $|\dot{\zeta}|$ will both be bounded by some ℓ_{\max} and v_{\max} respectively for all $t \leq T$.

As a result, for any $s, t \in [0, T]$ with $s < t$, we will also have

$$|\zeta(t) - \zeta(s)| \leq \int_s^t |\dot{\zeta}(\tau)| d\tau \leq v_{\max}(t - s), \quad (6.13)$$

so if $t_n \rightarrow T$ is Cauchy, the same will hold for $\zeta(t_n)$ as well; hence, with S closed, we will have $\lim_{t \rightarrow T} \zeta(t) \equiv \zeta_T \in S$. With $\dot{\zeta}$ bounded, we will then also have

$$|\dot{\zeta}(t) - \dot{\zeta}(s)| \leq \int_s^t |\ddot{\zeta}(\tau)| d\tau \leq F_{\max}(t - s) + \sum_{\beta} \int_s^t |N_{\beta}(\zeta(\tau), \dot{\zeta}(\tau))| d\tau, \quad (6.14)$$

and with $\sup |\zeta|, \sup |\dot{\zeta}| < \infty$, it follows that the components $|N_{\beta}|$ of the centripetal force will also be bounded: $x(t) = \Phi^{-1}(\zeta(t))$ remains a positive distance away from $\text{bd}(X)$ for all $t \leq T$, so the weight coefficients $\theta_{\beta}''' / (\theta_{\beta}'')^2$ of the centripetal force N in (6.8b) are bounded, and the same holds for the velocity components $\dot{\zeta}_{\beta}^2$. We will thus have $|\dot{\zeta}(t) - \dot{\zeta}(s)| \leq a(t - s)$ for some $a > 0$, so the limit $\lim_{t \rightarrow T} \dot{\zeta}(t)$ exists and is finite. In this way, if we take (E-ID) with initial conditions $\zeta(T) = \zeta_T$ and $\dot{\zeta}(T) = \lim_{t \rightarrow T} \dot{\zeta}(t)$, the Picard–Lindelöf theorem shows that the original maximal solution $\zeta(t)$ may be extended beyond the maximal integration time T , a contradiction.

For the converse implication, assume that S is not closed in \mathbb{R}^{n+1} , let \bar{S} denote its closure, and let $q \in \bar{S} \setminus \{S\}$. Clearly, \bar{S} will be a closed submanifold-with-boundary of \mathbb{R}^{n+1} and the metric induced by the inclusion $\bar{S} \hookrightarrow \mathbb{R}^{n+1}$ on \bar{S} will agree with the one induced by the inclusion $S \hookrightarrow \mathbb{R}^{n+1}$ on S . Thus, let $\gamma(t)$ be a geodesic of \bar{S} which starts at q with initial velocity v_0 pointing towards the interior of S , and let $T > 0$ be sufficiently small so that $\gamma(T) = p \in S^{\circ}$. Furthermore, let $v_T = \dot{\gamma}(T)$ be the velocity with which $\gamma(t)$ reaches p ; by the invariance of the geodesic equation with respect to time reflections, this means that the geodesic which starts at p with velocity $-v_T$ will reach q at finite time $T > 0$ with outward-pointing velocity $-v_0$. Noting that geodesics on S are simply solutions of (E-ID) for $u \equiv 0$ and $\eta = 0$, and carrying (E-ID) back to X° via the isometry (EC), we have shown that (ID) admits a solution which escapes from X° in finite time, i.e. (ID) fails to be well-posed if S is not closed.¹⁸ \square

¹⁸For general u and $\eta > 0$, simply let $\gamma(t)$ be a solution of the dynamics $\ddot{\zeta} = F + N + \eta\dot{\zeta}$, i.e. (E-ID) with $-\eta\dot{\zeta}$ replaced by the momentum-gathering term $\eta\dot{\zeta}$. The time-reflected variant of this equation is simply (E-ID), so the rest of the argument follows in the same way.

Remark 6.1. It is interesting to note that Theorem 6.1 relates an analytic property of the inertial dynamics (ID) on X° (their well-posedness) to a topological property of the geometry of X° (the closedness of the Nash–Kuiper embedding (EC) of X°). The asymptotic behavior of the inertial dynamics (ID) is thus tied to the underlying geometry of X° , an interdependence which does not exist in first order (see also Fig. 1): by contrast, Hessian–Riemannian gradient flows were shown in [4] to be well-posed independently of the underlying geometry.

From a more practical point of view, Theorem 6.1 allows us to verify the well-posedness condition (WP) simply by checking that the Euclidean image S of X° is closed. Indeed:

Corollary 6.2. *The inertial dynamics (ID)/(IGD) are well-posed if and only if the kernel θ of the Hessian–Riemannian structure of X satisfies $\int_0^1 \sqrt{\theta''(x)} dx = +\infty$.*

Proof. Simply note that the image $S = \Phi(X^\circ)$ of X° under (EC) is bounded (and hence, not closed) if and only if $\int_0^1 \phi'(x) dx = \int_0^1 \sqrt{\theta''(x)} dx < +\infty$. \square

In light of the above, we finally see that the analysis of Sections 4 and 5 is not vacuous:

Corollary 6.3. *The dynamics (I-LD) and the associated game dynamics (IGD) for the log-barrier kernel $\theta_L(x) = -\log x$ are well-posed.*

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